5

Advances in Water Resources Vol. 1 No. 4 1978

Harbour Resonance Problems Using Finite Elements

S. WALKER and C. A. BREBBIA

Southampton University

Received January 1978 Accepted February 1978

The problem of harbour resonance is discussed, and the probabilistic analysis of a harbour's response is carried out using a finite element technique.
The results presented in this paper show the importance of considering two types of damping (radiation and friction) in the study of harbour resonance phenomena. The radiation, or energy transfer between the harbour and the open sea, is introduced using an expression similar to Sommerfeld's radiation condition, and appears as a boundary only damping term.
The three analyses with or without damping and with or without radiation are compared, and it is shown that it is important to include both effects to avoid distortion of the response spectrum. The probability of exceedance for elevations is calculated for the Duncan Basin, and a way of calculating the horizontal displacements is indicated.

1. INTRODUCTION

Many harbours are limited in their usefulness by seiching associated with excitation at one or more of their natural frequencies. For harbours of regular shapes the natural frequencies can be found by analytical techniques, otherwise one needs to undertake expensive model experiments or turn to numerical techniques. The finite element method is well suited to the study of harbours of arbitrary dimensions, as the size (and depth) of elements can be varied at will. The main difficulty associated with this method is the proper representation of the mechanism of energy exchange between the harbour and the sea. This representation is done here by applying on the boundary a radiation condition that allows the wave energy to be transferred without distortion. In addition wave reflection due to the surrounding coastline can be incorporated into the model. Most theoretical models used up to now have concerned themselves with predicting the natural periods of the water body, but have ignored the effect of friction between the bed and the moving fluid. In order to obtain realistic estimates of displacements the friction effects need to be taken into account.

The usual design procedure is to estimate the likely period of waves in the locality and then analyse the response of the system. The designer tries to mismatch the frequencies of the incoming waves and the fundamental frequency of the system. This assumption of a deterministic design wave is in fact a drastic oversimplification of the real situation. Waves are essentially random in nature and this necessitates a probabilistic approach. The wave record at a point is, in fact, the superposition of an infinite number of harmonic components of different amplitudes and frequencies. This record is normally represented by a wave spectrum for the location which will be peaked at the dominant period of the waves. These recorded spectra which vary from region to region of the sea according to the local characteristics of the waves, are then to be used as input for the statistical analysis.

The statistical analysis is based on a transfer function given by the system response to simple harmonic wave excitations. The transfer function converts the wave spectrum into a response spectrum of wave heights (such as those inside a harbour). Integration of the response spectrum with respect to the frequencies gives the variance of the variable under consideration and this allows one to predict the probability of a maximum value being exceeded. In addition, by using the momentum equations, the horizontal displacement spectra can be deduced from the wave height spectrum.

Numerical results were obtained for the case of the old Duncan Basin in South Africa. This harbour has been extensively studied and the shape of the Bay is such that some of the frequencies are greatly amplified, producing large oscillations. The Basin was discretized using a finite element mesh and the radiation condition applied on the part of the boundary representing the harbour mouth. The program was run to study the effect of including friction and radiation on the results. As the radiation condition allows energy to escape from the harbour, it has the effect of reducing the peaks of the lowest frequencies, especially for nodes near the harbour mouth. The inclusion of friction however, has the effect of reducing the amplitudes at the higher frequencies.

The numerical results show that it is essential to include both effects in the analysis in order to obtain a representation of the harbour's behaviour.

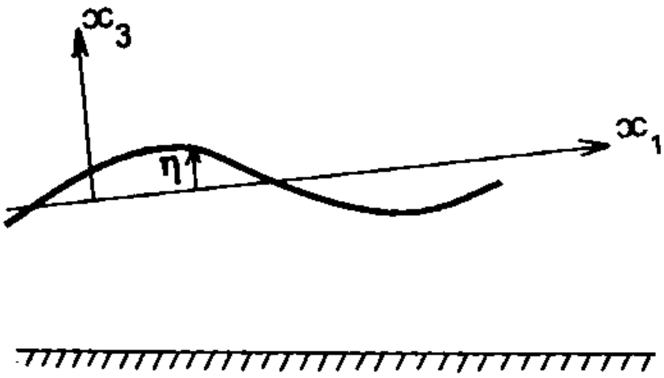

Figure 1 Geometrical Definitions

2. GOVERNING EQUATIONS

The harbour resonance equation for harmonic motion can be written as:

$$\frac{\partial}{\partial x_1}\left(h\frac{\partial \eta}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(h\frac{\partial \eta}{\partial x_2}\right) + \frac{\omega^2}{g}\eta = 0, \tag{1}$$

with boundary condition

$$h\frac{\partial \eta}{\partial n} = \frac{i\omega}{g}\,\bar{q}_n \text{ on } S_2 \tag{2}$$

where η is the wave elevation referred to the still water level with in plane coordinates x_1 and x_2. h is the depth, g the acceleration due to gravity, $\bar{q}_n$ is the influx through S_2 boundaries.

In addition to boundary condition (2) one needs to apply the radiation condition on a boundary between the harbour and the sea.

η can now be split into 3 parts (Figure 2),

(i) The incident field η_I.

(ii) The reflected field calculated from plane wave reflections by the boundaries and called η_R (see Figure 2, boundaries AG and CE).

(iii) The scattered field η_S originating from inside the harbour.

Hence the total wave field is

$$\eta = \eta_I + \eta_R + \eta_S \tag{3}$$

with the incident field given by

$$\eta_I = a\, e^{-i\kappa r \cos(\theta - \alpha)} \tag{4}$$

κ is the wave number which for shallow water is,

$$\omega^2 = g\,\kappa^2\, h \rightarrow \kappa^2 = \omega^2/gh \tag{5}$$

where h is a representative depth along the contour where the radiation condition is to be applied. The radiation condition one uses on the sea boundary S_1 is now

$$\frac{\partial \eta_S}{\partial r} + i\kappa\eta_S = 0. \tag{6}$$

This gives

$$\frac{\partial \eta}{\partial r} + i\kappa\eta = \underbrace{\frac{\partial}{\partial r}(\eta_I + \eta_R) + i\kappa(\eta_I + \eta_R)}_{\text{known}} = f \tag{7}$$

The r.h.s. of this equation is a known function of the horizontal coordinates, the angles determining the local geometry and the angle of incidence of the wave field η_I. On the land boundaries S_2 with no flux we have $\partial\eta/\partial n = 0$. For the harbour oscillation problem the geometry and boundary conditions are shown in Figure 2. For this case n = r.

Conditions (2b) and (6) together with the governing equation (1) can be written in the following Galerkin's type statement,

$$\iint\left\{\frac{\partial}{\partial x_1}\left(h\frac{\partial \eta}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(h\frac{\partial \eta}{\partial x_2}\right) + \kappa^2 h\,\eta\right\}\delta\eta\, dx_1\, dx_2 = \int_{S_2}\left\{h\frac{\partial \eta}{\partial n} - \bar{q}_n\right\}\delta\eta\, dS + \int_{S_1} h\left\{\frac{\partial \eta}{\partial n} + i\kappa\eta - f\right\}\delta\eta\, dS. \tag{8}$$

Integrating by parts,

$$\iint\left\{h\frac{\partial \eta}{\partial x_1}\frac{\partial \delta\eta}{\partial x_1} + h\frac{\partial \eta}{\partial x_2}\frac{\partial \delta\eta}{\partial x_2} - \kappa^2\, h\,\eta\delta\eta\right\}dx_1\, dx_2 = \int_{S_2}\bar{q}_n\,\delta\eta\, dS + \int_{S_1} hf\,\delta\eta\, dS - \int_{S_1} ih\,\kappa\eta\,\delta\eta\, dS. \tag{9}$$

For the case $\bar{q}_n = 0$, we can write,

$$\iint\left\{h\frac{\partial \eta}{\partial x_1}\frac{\partial \delta\eta}{\partial x_1} + h\frac{\partial \eta}{\partial x_2}\frac{\partial \delta\eta}{\partial x_2} - \kappa^2\, h\eta\,\delta\eta\right\}dx_1\, dx_2 + \int_{S_1} ih\,\kappa\eta\,\delta\eta\, dS = \int_{S_1} hf\,\delta\eta\, dS \tag{10}$$

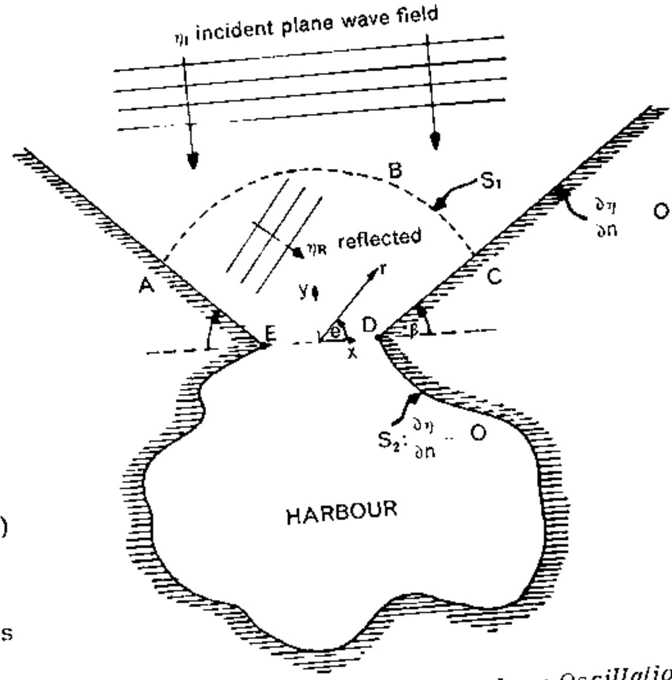

Figure 2 Definition Sketch for Harbour Oscillatio Problem

We now assume that the variable u can be approximated on each element by

$$\eta = \underset{\sim}{\mathrm{N}}^{\mathrm{T}}\underset{\sim}{\mathrm{H}} \tag{11}$$

where $\underset{\sim}{\mathrm{N}}$ is an interpolation function vector and $\underset{\sim}{\mathrm{H}}$ are the nodal unknowns. Substituting (11) into (10) we obtain the following expression for an element

$$\delta\underset{\sim}{\mathrm{H}}^{\mathrm{T}}\left\{\iint \mathrm{h}\left(\frac{\partial\underset{\sim}{\mathrm{N}}}{\partial x_1}\frac{\partial\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\partial x_1} + \frac{\partial\underset{\sim}{\mathrm{N}}}{\partial x_2}\frac{\partial\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\partial x_2} - \kappa^2\,\underset{\sim}{\mathrm{N}}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\right)\mathrm{d}x_1\,\mathrm{d}x_2\,.\right.$$

$$\left. + \mathrm{i}\int_{S_1}\mathrm{h}\kappa\,\underset{\sim}{\mathrm{N}}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\,\mathrm{dS}\right\}\underset{\sim}{\mathrm{H}} = \delta\underset{\sim}{\mathrm{H}}^{\mathrm{T}}\left\{\int_{S_1}\mathrm{h}\underset{\sim}{\mathrm{N}}\,\mathrm{fdS}\right\}. \tag{12}$$

This expression can be written as,

$$\{\underset{\sim}{\mathrm{K}} - \kappa^2\,\underset{\sim}{\mathrm{M}} + \mathrm{i}\kappa\,\underset{\sim}{\mathrm{M}}'\}\underset{\sim}{\mathrm{H}} = \underset{\sim}{\mathrm{F}} \tag{13}$$

where

$$\underset{\sim}{\mathrm{K}} = \iint \mathrm{h}\left(\frac{\partial\underset{\sim}{\mathrm{N}}}{\partial x}\frac{\partial\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\partial x} + \frac{\partial\underset{\sim}{\mathrm{N}}}{\partial y}\frac{\partial\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\partial y}\right)\mathrm{d}x_1\,\mathrm{d}x_2$$

$$\underset{\sim}{\mathrm{M}} = \iint \mathrm{h}\underset{\sim}{\mathrm{N}}\,\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \underset{\sim}{\mathrm{M}}' = \int_{S_1}\mathrm{h}\underset{\sim}{\mathrm{N}}\,\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\,\mathrm{dS} \tag{14}$$

$$\underset{\sim}{\mathrm{F}} = \int_{S_1}\mathrm{h}\,\underset{\sim}{\mathrm{N}}\,\mathrm{f}\,\mathrm{dS}.$$

We can now assemble the different matrices defined by (13) into the global matrices for the whole continuum. [See reference 3.] This gives

$$\{\underset{\sim}{K} - \kappa^2\,\underset{\sim}{M} + \mathrm{i}\kappa\,\underset{\sim}{M}'\}\underset{\sim}{H} = \underset{\sim}{F}. \tag{15}$$

It can now be seen that the radiation condition manifests itself as a damping term on the external boundary.

For problems such as harbour resonance it is possible to include an extra term representing frictional effects inside the finite element domain. The linearized version gives the matrix equation

$$\{\underset{\sim}{K} - (\kappa^2 - \mathrm{i}\gamma\kappa)\underset{\sim}{M} + \mathrm{i}\kappa\,\underset{\sim}{M}'\}\underset{\sim}{H} = \underset{\sim}{F} \tag{16}$$

where γ is an empirical coefficient.

γ can be related to the Chezy coefficient through the momentum equations. This gives,

$$\gamma = \frac{\lambda}{\mathrm{gh}} = \frac{1}{\mathrm{hc}^2}|\mathrm{v}| \tag{17}$$

where $|\mathrm{v}|$ is the magnitude of the velocity, c is the Chezy coefficient and h is the water depth. In order to approximate γ one needs to find the time and space average values of the ratio $|\mathrm{v}|/\mathrm{h}$.

We may write the inverse of the multiplier of the nodal unknowns vector $\underset{\sim}{H}$ as $\underset{\sim}{A}(\kappa)$ to obtain formally

$$\underset{\sim}{H} = \underset{\sim}{A}(\kappa)\underset{\sim}{F} \tag{18}$$

where $\underset{\sim}{A}$ may now be thought of as a complex transfer function and

$$\underset{\sim}{A}(\kappa) = \{\underset{\sim}{K} - (\kappa^2 - \mathrm{i}\gamma\kappa)\underset{\sim}{M} + \mathrm{i}\kappa\,\underset{\sim}{M}'\}^{-1}. \tag{19}$$

Equation (18) gives the response of the system to a harmonic excitation. At a point 'i' the displacement can be written as

$$\underset{\sim}{\mathrm{H}}_{\mathrm{i}} = \underset{\sim}{A}_{\mathrm{i}}\,\underset{\sim}{F} \tag{20}$$

where $\underset{\sim}{A}_{\mathrm{i}}$ is the ith row of the $\underset{\sim}{A}$ matrix.

3. SPECTRAL ANALYSIS

So far we have assumed that the incoming wave is harmonic with given period and amplitude. The problem is then one of a simple forced vibration and the solution is straightforward. Unfortunately, this is not what happens in reality where the wave record is essentially the superposition of harmonic waves of differing amplitudes with random phase. The wave field is most commonly represented by the wave spectrum for the location. Figure 3 shows a typical plot of wave height spectral densities $S_{FF}(\omega)$ against angular frequency ω in radians per second.

If F(t) is a randomly varying quantity with zero mean then a basic property of the spectrum is that the mean squared value of F(t) or variance σ^2, is given by

$$\sigma^2 = \langle F^2(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}F^2(t)\mathrm{d}t = \int_0^{\infty}S_{FF}(\omega)\mathrm{d}\omega. \tag{21}$$

$S_{FF}(\omega)$ represents the way in which the harmonic components of F are distributed over the frequency range. The quantity of $\langle F^2(t)\rangle$ associated with a narrow frequency interval $[\omega, \omega + \Delta\omega)$ is simply $S_{FF}(\omega)\Delta\omega$.

It is clear that the response at a given point will be a random quantity and will in turn be described by a response spectrum $S_{HH}(\omega)$. To find the relationship between S_{FF} and S_{HH} let us first define a forcing vector $\underset{\sim}{F}_0$ representing a unit actual value of the function which will be called F. Hence, the height at a point 'i' can be written as

$$H_i = \alpha_i(\omega)F. \tag{22}$$

(For instance F may be the incident wave height for a harbour.)

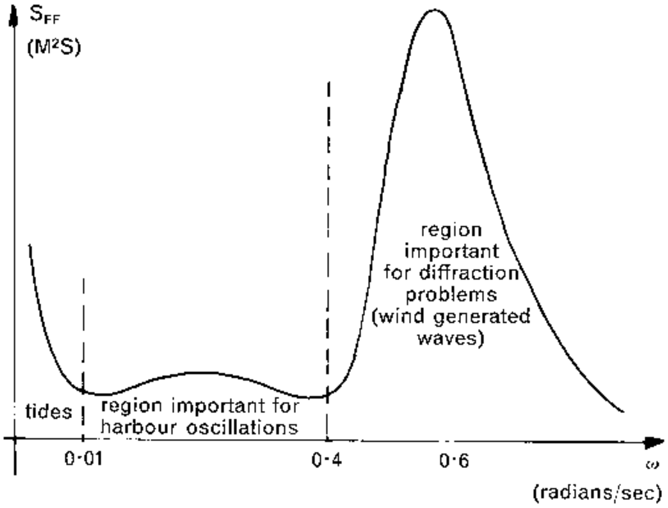

Figure 3 Typical Wave Spectrum (Logarithmic Scales)

Note that

$$\alpha_i(\omega) = \underline{A}_i^T(\omega)\underline{F}_0 . \tag{23}$$

To find the relationship between S_{FF} and S_{HH} we can compute the deviation of H and F by squaring equation (22). Taking into account the fact that H_i is a complex number.

Hence, we can write for a point 'i'

$$H_i^* \, H_i = \alpha_i^* \, F^* \, \alpha_i \tag{24}$$

where the asterisk denotes complex conjugation.

Dividing equation (24) by T, the length of the record and letting $T \to \infty$ gives

$$S_{H_iH_i}(\omega) = |\alpha_i(\omega)|^2 \, S_{FF}(\omega) \tag{25}$$

as

$$S_{H_iH_i}(\omega) = \lim_{T\to\infty} \left\{ \frac{1}{T} H_i^* \, H_i \right\}$$

and

$$S_{FF}(\omega) = \lim_{T\to\infty} \left\{ \frac{1}{T} F^* \, F \right\} .$$

See reference [2].

Equation (25) gives the relationship between the spectral density input function S_{FF} and the response spectrum $S_{H_iH_i}$. We can see that any particular ordinate of the $S_{H_iH_i}$ spectrum can be obtained by multiplying the corresponding ordinate by the modulus squared of the α_i function at that frequency. The ordinates of α_i are in fact the magnitude of the response at the point 'i' due to a unit amplitude forcing function for a given frequency ω. The values of α_i can be obtained using finite elements. Having obtained the spectrum of response we can integrate the area under it to obtain the mean squared value or variance σ^2. Note that

$$\sigma_H^2 = \langle H^2 \rangle = \int_0^\infty S_{HH} \, d\omega \tag{26}$$

for a process with zero mean. Furthermore, if the process is Gaussian or broad banded we can assign probabilities for the response exceeding various multiples -μ- of the standard deviation σ, i.e. the probability of the heights being within a certain $\pm\mu\sigma$ value is given in the following table.

Table 1

μ	Probability of $-\mu\sigma \leq \eta \leq \mu\sigma$	Probability of $\lvert\eta\rvert > \mu\sigma$
1	68.3%	31.7%
2	95.4%	4.6%
3	99.7%	0.3%

Alternatively if the input process is narrow banded (i.e., a storm or tsunami) we have a Rayleigh distribution, i.e.

$$P(H) = \frac{H}{\sigma_H^2} \exp \left\{ -H^2/2\sigma_H^2 \right\} \tag{27}$$

for a given storm.

The expected maximum value of the response can be approximated by,

$$\langle |H|_{max} \rangle = \sigma_H \left\{ \sqrt{2 \ln(T/T_m)} \right\} \tag{28}$$

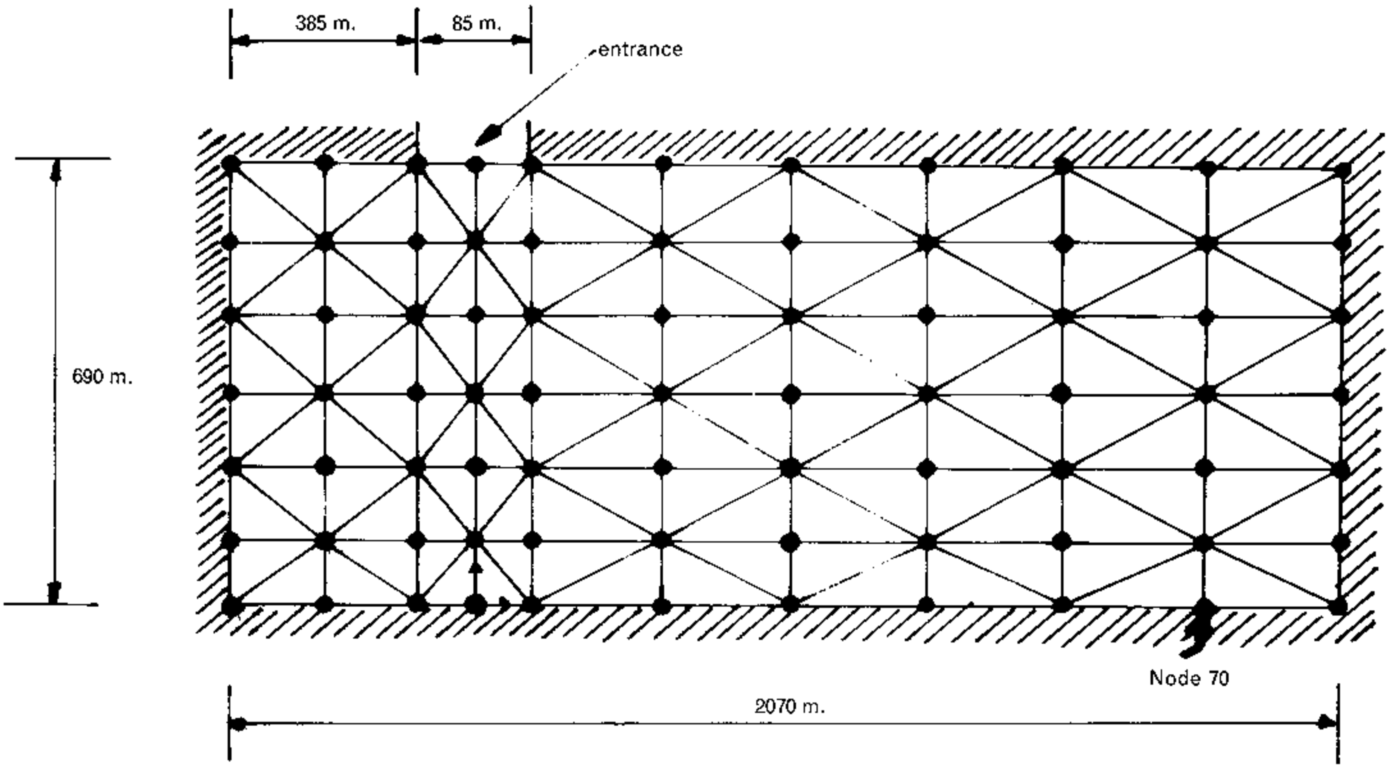

Figure 4 Mesh for Duncan Basin

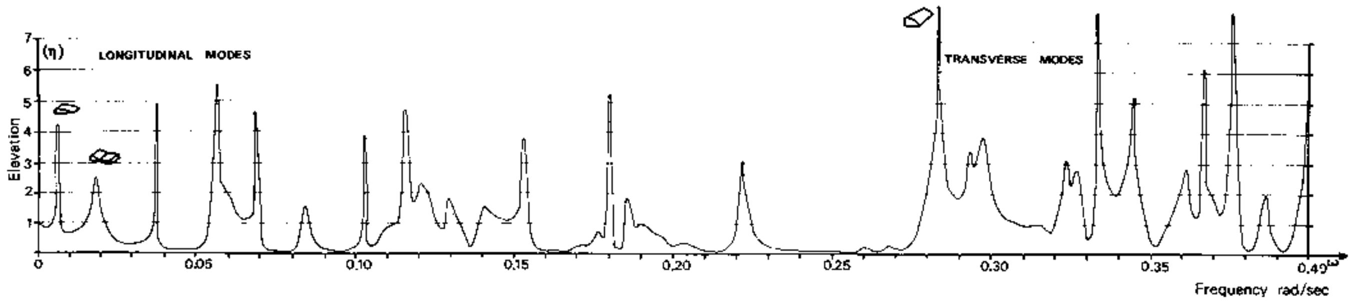

Figure 5 Maximum Surface Elevation at node 70 with Radiation but without Damping

where T_m is the mean period, given by

$$T_m = 2\pi \left(\frac{\int_0^\infty S_{HH}\, d\omega}{\int_0^\infty \omega^2 S_{HH}\, d\omega} \right)^{1/2} . \tag{29}$$

If the process is not narrow banded one can still numerically calculate its probability and expected maximum but the Rayleigh distribution is no longer valid.

EXAMPLE FOR HARBOUR OSCILLATIONS

Numerical results with, and without the use of the radiation condition were obtained for the case of the old Duncan Basin within Table Bay in South Africa. This harbour has been extensively studied [4], [5] and the shape of the Bay is such that some of the frequencies are greatly amplified, producing large oscillations within the Basin. This fact has been demonstrated by model experiments, harmonic analyses, seichograms and simple theoretical analyses, which given reasonable results here as the shape of the basin is approximately rectangular. The exact natural periods for a rectangular closed basin of constant depth are given by

$$T_{mn} = \frac{2l}{\sqrt{gd}\,(m^2 + \beta^2 n^2)^{1/2}} , \beta = \frac{l}{b} \tag{30}$$

where m and n describe the modal shapes in the two directions (m is associated with the length of the basin l and n with the width b).

To simulate deterministically the effect of waves entering the harbour we discretised it into finite elements (figure 4). Along the contour BCDEFA we applied the usual zero normal flux condition $\partial\eta/\partial n = 0$. On the harbour entrance AB we applied the radiation condition for an incident plane wave of unit amplitude entering the harbour in the negative y direction, for the moment ignoring reflections from the neighbouring coastline. Damping was not included in this part of the study. The matrix equation to be solved in this case is

$$\{\underline{K} - \kappa^2 \underline{M} + i\kappa \underline{M}'\}\underline{H} = \underline{F}. \tag{31}$$

Solving equation (31) for $\underline{H}$ gives a vector of the displacements in complex form at each node. Once the deterministic response is found it can be squared at the points under consideration to give the probabilistic transfer function for the resulting wave elevation at each node resulting from a wave of unit amplitude incident at the harbour mouth. One advantage of using the radiation condition is that the *direction* of this incident wave may be varied easily. The resulting elevations at node 70 for unit incident waves of differing frequencies are plotted in figure 5, the transfer function is given by the squares of these values. The first peaks occur at the same frequencies as those predicted by the harmonic analysis given in reference [6]. For instance the first peak represents the first significant period of about 11.45 minutes which is the mode corresponding to water flowing in and out of the basin and is called the 'pumping mode'. The second peak also corresponds to a simple motion the so-called 'sloshing mode'. The results were plotted up to $\omega = 0.4$ radians/second which is the lower bound of the wind generated part of the wave spectrum.

It is interesting to note that after a series of mainly longitudinal modes, the transverse modes start to play a more significant part. The effect of these modes is not at all negligible, especially from the probabilistic response point of view. The application of the radiation boundary condition on the harbour mouth adequately represents the input of wave energy into the system as well as energy passing *out* of the harbour.

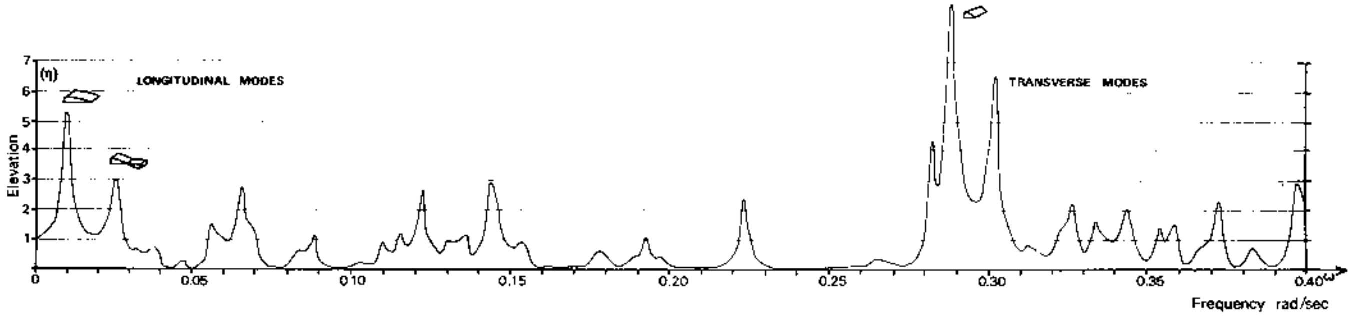

Figure 6 Maximum Surface Elevation at node 70 without Radiation with Damping $\gamma = 0.0002$

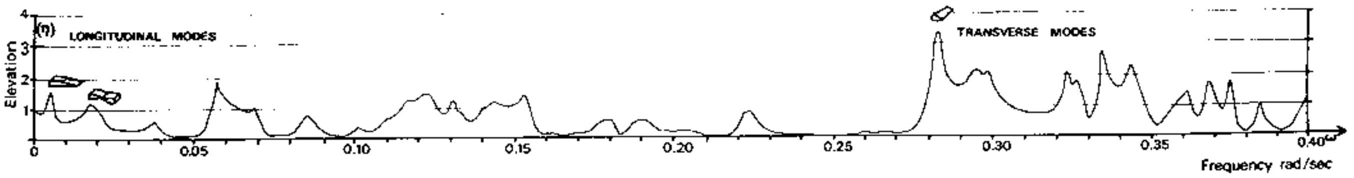

Figure 7 Maximum Surface Elevation at node 70 including both Radiation and Damping effects $\gamma = 0.0002$

Another study was carried out on the same problem region without the use of the radiation condition but including the damping term representative of bottom friction. In this case the wave elevation at the entrance nodes was fixed equal to the amplitude of the incoming wave. The matrix equation to be solved is now

$$\{\underset{\sim}{K} - (\kappa^2 - i\kappa\gamma)\underset{\sim}{M}\}\underset{\sim}{H} = \underset{\sim}{P}. \tag{32}$$

The elements of the $\underset{\sim}{P}$ vector are formed by multiplying the known elevation values at the harbour mouth by elements of the original left hand side and transferring the result to the right hand side. The transfer function was obtained by squaring the elements of the response vector $\underset{\sim}{H}$ in the usual way.

The wave height responses at node 70 are shown in figure 6. Again the first significant modes appear at the expected frequencies.

The value of $\gamma (= 0.0002)$ was taken to be representative of bottom friction only. It is interesting to note that the first longitudinal and transverse modes seem to dominate the motion, the others are quickly damped. (see references 7, 8 and 9).

Finally the wave height responses were calculated for the harbour including both the effects of bottom friction and transfer of energy across the harbour mouth by the use of the radiation condition.

The matrix equation to be solved in this case is

$$\{\underset{\sim}{K} - (\kappa^2 - i\kappa\gamma)\underset{\sim}{M} + i\kappa\,\underset{\sim}{M'}\}\underset{\sim}{H} = \underset{\sim}{F} \tag{33}$$

where the matrices are defined in equations 14. The incident wave is of unit amplitude entering the harbour in the negative y direction and the resulting elevations at node 70 are plotted in figure 7. The peaks appear in the expected positions, and the spurious peaks which appeared at high frequencies in the first study have been significantly reduced by the inclusion of internal damping.

The natural frequencies of harbours generally lie in the region between wind and tides (i.e. approximately between frequencies ω of 0.01 to 0.40 radians/second). Usually the energy of the oceans in this region is not considerable, but some phenomena can produce agitation which will damage the ships moored in harbours. Large displacements of the water occur for relatively small vertical motion.

Unfortunately, the spectrum corresponding to the Duncan Basin was not available, consequently it was decided to use a white noise type spectrum with a cut-off at 0.40 radians/second and a magnitude of 0.01 m^2s. For this study friction was considered in the form shown in equation (32). The value of the constant γ was taken to be 0.0002.

Multiplying the square of the ordinates of the responses, by the ordinates of the (white noise) sea spectrum we obtained the response spectra for the three cases for the point under consideration. By integrating the areas under the final figures we obtained the variances of the displacements. The values obtained for the variances in the three cases are shown in Table II. These values indicate the importance of the transverse modes' contribution to the probabilistic response.

As can be expected from probability theory the variances can be added, i.e.

$$\sigma_R^2 - \sigma_D^2 \approx \sigma_{RD}^2. \tag{33}$$

For the white noise spectrum, $S_{\eta\eta}(\omega) \equiv 0.01\ m^2sec$, we obtained the following results at the 99.7% confidence level.

	SURFACE ELEVATION		
	$S_{\eta\eta} \times \sigma^2$	Standard Deviations	Maximum = $3 \times \sigma$
Radiation only	0.013105	0.11447	0.343 M
Damping only	0.009039	0.09507	0.285 M
Radiation and Damping	0.003319	0.05761	0.1728 M

Table 2 Variances of surface elevation at point 70

Range of Frequencies	Radiation Only	Damping Only	Radiation and Damping	Comments
$\omega = 0$ to 0.27	0.3475	0.4475	0.0896	Contributions from Mainly Longitudinal Modes
$\omega = 0.27$ to 0.4	0.9632	0.7564	0.2424	Contribution from Mainly Transverse Modes
TOTAL	$\sigma_R^2 = 1.3105$	$\sigma_D^2 = 0.9039$	$\sigma_{RD}^2 = 0.3319$	Total variance

The horizontal displacement spectrum can be obtained from the momentum equations. For harmonic motion they give

$$V_j = \frac{1}{\kappa^2 - i\kappa\gamma}\frac{\partial H}{\partial x_j} \qquad j = 1, 2, \ldots \tag{34}$$

where V_j is a maximum displacement such that

$$v_j = V_j \, e^{i\omega t}$$

Hence the displacement spectrum for a point i will be

$$S_{v_i v_i} = \frac{1}{\kappa^4 + \kappa^2\gamma^2} S_{H,\,j^{H},\,j} \tag{35}$$

where , j denotes the derivative $\partial/\partial x_j$.

Once the spectrum of derivatives of wave height in the given direction is known, the variance of the displacements in that particular direction can be calculated by integration. Again we can define the maximum displacements which will be in the range $\pm 3\sigma_v$ within a certainty of 99.7%.

CONCLUSIONS

The results presented in this paper show the importance of considering the two types of damping (radiation and friction) in the study of harbour resonance phenomena. The radiation which appears as a boundary only damping tends to reduce the peaks of lower frequencies for each mode of oscillation. The internal damping which appears throughout the region tends to reduce the higher frequency peaks.

In practice it is usual to increase the internal damping coefficient to take account of the energy loss through radiation. This approach, however, tends to distort the shape of the response curves and give expected responses which are too low.

Probabilistic solutions are more realistic from the designer's point of view and can be easily obtained once the response of the system is known. As an illustration the probability of exceedance for elevations were calculated for the Duncan Basin, and a way of calculating the horizontal displacements was indicated. Note that small elevations give rise to large horizontal displacements which are critical for the mooring of ships.

REFERENCES AND BIBLIOGRAPHY

1. Sommerfeld, A. *'Partial Differential Equations in Physics'* New York, Academic Press 1949 (pp. 256-344).
2. Walker, S. *'Determination of the Wave Forces on Offshore Gravity Platforms, by Diffraction Theory and Spectral Analysis'* University of Southampton, Department of Civil Engineering, Technical Report C.H. 1/77.
3. Brebbia, C. A. and Connor, J. J. *'Introduction to Finite Element Techniques for Fluid Flow'*, Butterworths, London, 1976.
4. Wilson, B. W. *'The Mechanisms of Seiches in Table Bay Harbor, Cape Town'*, Proc. 4th Conf. on Coastal Eng'g, 1953, Published by Council in Wave Research, 1954.
5. Wilson, B. W. 'Research and Model Studies on Wave Action in Table Bay Harbor, Cape Town', *Transactions of the South African Institution of Civil Engineers,* Vol. 1, No. 6 and 7, Vol. 2, No. 5, June, July, 1959, May, 1960.
6. Brebbia, C. A. and Adey, R. *'Circulation Problems'*, Proc. Finite Element Symposium, Atlas Computer Laboratory, Didcot, U.K. March 26-28, 1974.
7. Dorrenstein, R. *'Amplification of Long Waves in Bays'*, Engineering Progress at the University of Florida Bulletin, Vol. 15, No. 12, 1961.
8. Harleman, D. R. F. and Ippen, A. T. *'Friction and Energy dissipation on Long Wave Hydrodynamics'* Proceedings of the Tsunami Meetings Associated with the 10th Pacific Science Congress. International Union of Geodesy and Geophysics. Hawaii, August-September, 1961.
9. Shaw, R. P. and Chi-Kwong Lai, *'Channel Friction and Shape Effects on Harbor Resonance'*, Proceedings A.S.C.E., Waterways and Harbors Div. Journal, August, 1974.
10. Raichlen, F. *'Harbour Resonance'*, 'Estuary and Coastline Hydrodynamics', Edited by A. T. Ippen, Chapter 7, McGraw Hill, 1976.
11. Lee, J. J. and Raichlen, A. M. *'Oscillations in Harbors with Connected Basins'*, Proc. A.S.C.E., Waterways, Harbors and Coastal Eng. Div., August, 1972.
12. Ippen, A. T. and Fattah, Q. M., *'Wave Induced Oscillation in Harbors: The Effect of the Variation of Geometric Parameters on the Response of Rectangular Harbors'*, Ralph Parson Laboratory, Report No. 70, M.I.T. June, 1964.
13. Ippen, A. T. and Goda, Y. *'Wave Induced Oscillations in Harbors. The Solution for a Rectangular Harbor Connected to the Open Sea'*, M.I.T. Hydrodynamics Lab. Report No. 59, July, 1963.
14. Eagleson, P. S. et al., *'Hydraulic Model Study of Protective Works for Fleet Berthing Facilities in Coddington Cave, Narragansett Bay, R.I.'*. M.I.T. Hydrodynamics Lab., Report No. 25, June, 1957.
15. Wilson, B. W., Hendrickson, J. A. and Len, J. *'Surge in the Southeast Basin, Long Beach Harbor, California'*, Proc. 13th Conf. on Coastal Eng., Vol. 3, A.S.C.E., 1972.
16. Miles, J. and Munk, W. *'Harbor Paradox'*, Journal of the Waterways and Harbors Division, A.S.C.E., Vol. 87, No. WW3, Proc. Paper 2888, August, 1961, pp. 111-130.
17. Lee, J. J. *'Wave-Induced Oscillations in Harbors of Arbitrary Geometry'*, Journal of Fluid Mechanics, Vol. 45, 1971, pp. 375-394.
18. Lee, J. J. and Raichlen, F. *'Resonance in Harbors of Arbitrary Shape'* Proceedings of the 12th Conference on Coastal Engineering, Washington, D.C., 1970, Chapter 131, pp. 2163-2180.
19. Chen, H. S. and Mei, C. C. *'Oscillation and Wave Forces in a Man-Made Harbor in the Open Sea'*, Tenth Symposium on Naval Hydrodynamics Office of Naval Research, 1975.
20. Heaps, N. S. *'Tidal Effects due to Water Power Generation in the Bristol Channel'*, Tidal Power (edited by T. J. Gray and O. K. Geshus), Plenum Press, N.Y. 1972.
21. Davenport, A. G. *'Note on the Distribution of the Largest Value of a Random Function with Applications to Gust Loading'*, Proc. Inst. Civil Eng'g, Paper No. 6739, 1964.
22. Kravtchenko, J. and McNown, J. S. *'Seiche in Rectangular Ports'*, Quarterly of Applied Mathematics, Vol. 13, 1955, pp. 19-26.