Simplified boundary elements for radiation problems

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Introduction

Radiation boundary conditions need to be taken into account in many engineering problems in order to obtain realistic results. These conditions can be modelled using the fundamental solution as sources on the boundaries. This approach, however, presents difficulties for cases such as the two-dimensional Helmholtz equation where the fundamental solution is expressed in terms of a Hankel function. In addition, the source formulation produces interaction coefficients such that every boundary node is related to all the others, which destroys the banded character of the solution (usually finite elements) inside the domain.

This note describes a simple way of forming approximate boundary elements. These simplified elements are easy to combine with finite elements and do not give full matrices. The elements are used for solving the Helmholtz equation, but the basic idea can be applied to a wide range of problems. They are simple to implement in existing finite element packages and economic to use. The method is described for the two-dimensional domain but it is equally applicable to three-dimensional problems.

Formulation

Consider the problem depicted in *Figure 1* for which the following Helmholtz equation applies in region Ω_2 away from sources and solid bodies:

$$\nabla^2 u + \kappa^2 u = 0 \tag{1}$$

where u is the potential and κ the wave number. It is assumed that the domain comprises two regions; region Ω_1 which has been decomposed into finite elements and region Ω_2 which extends to infinity. They join at the interface Γ. The fundamental solution for equation (1) u^* is given by:

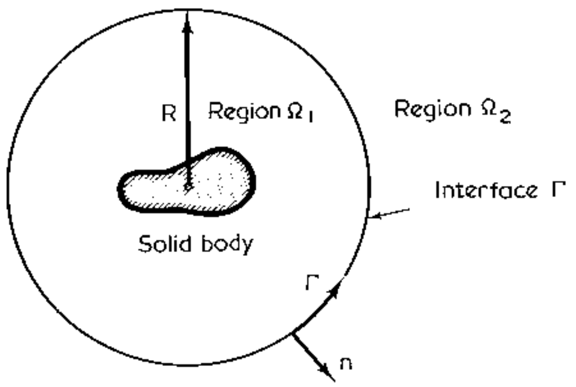

Figure 1 Domain under consideration

$$u^* = \frac{1}{4i} H_0^{(1)}(\kappa r) \tag{2}$$

where r is the distance from the source point (usually on the boundary) to the point of observation.

Weighting equation (1) with (2) we obtain[5]:

$$\int_\Omega (\nabla^2 u + \kappa^2 u) u^* \, d\Omega = 0 \tag{3}$$

Integrating this equation by parts twice gives:

$$\int_\Omega (\nabla^2 u^* + \kappa^2 u^*) u \, d\Omega = \int_\Gamma \frac{\partial u^*}{\partial n} u \, d\Gamma - \int_\Gamma u^* \frac{\partial u}{\partial n} d\Gamma \tag{4}$$

Depending on the position of the observation point, so far we have[3]:

$$\int_\Gamma \left(u^* \frac{\partial u}{\partial n} - \frac{\partial u^*}{\partial n} u \right) d\Gamma = \begin{matrix} u & \text{in} & \Omega_1 \\ \tfrac{1}{2}u & \text{on} & \Gamma \\ 0 & \text{in} & \Omega_2 \end{matrix} \tag{5}$$

For the fundamental solution of the Helmholtz equation, the left hand side of equation (5) is:

$$\int_\Gamma \left\{ H_0^{(1)}(\kappa r) \frac{\partial u}{\partial n} - \frac{\partial H_0^{(1)}(\kappa r)}{\partial n} u \right\} d\Gamma \tag{6}$$

We could now work with these functions, this has been done by Mei[1] and by more recent workers[2,3,4]. This approach has the disadvantage that the boundary values are all interrelated, resulting in fully populated matrices.

In the conventional boundary element approach the observation point is taken to be on the boundary. Each observation point gives one equation that connects the unknowns $\partial u/\partial n$ and u, which are evaluated on the boundary. If, however, we take the observation points to be in the region Ω_2, far from our problem region, we may make the following important simplification, using the asymptotic forms for the Hankel function, hence:

$$H_0^{(1)}(\kappa r) \simeq \sqrt{\frac{2}{\pi \kappa r}} e^{i(\kappa r - \pi/4)} \tag{7}$$

$$\frac{\partial H_0^{(1)}(\kappa r)}{\partial n} = -\kappa H_1^{(1)}(\kappa r) \simeq i\kappa \sqrt{\frac{2}{\pi \kappa r}} e^{i(\kappa r - \pi/4)}$$

(note that $n \simeq r$).

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...ll for large r: ...nto equation (5) for Ω_2 we

$$\int_{\Gamma} \left\{ \frac{\partial u}{\partial n} - i\kappa u \right\} \mathrm{d}\Gamma = 0 \tag{8}$$

We have obtained the radiation or Sommerfeld condition:

$$\frac{\partial u}{\partial n} = i\kappa u \tag{9}$$

This condition has been applied in many fluid mechanics problems, and works well even for complex problems[5].

The present approach is, however, more general than the simple use of the Sommerfeld condition as it can be extended to a wide range of problems, which involve different governing equations. Once the condition is established, boundary elements can be formulated as indicated elsewhere[6]. These elements can be easily implemented in existing finite element computer programs.

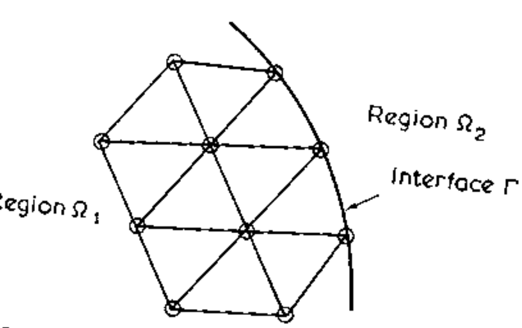
Applications

To determine how the application of the above radiation boundary condition affects the accuracy of the solution, the case of a single, vertical column subject to an incident harmonic wave was studied.

The column was surrounded by a finite element mesh as shown in *Figure 4*, and wave diffraction results were found for different wave lengths, with the condition given in equation (9) applied on the external boundary.

To test the adequacy of the radiation condition for representing a train of plane harmonic waves, the case with no solid cylinder was studied. For long waves (λ wave length $\simeq$ 30 m) the results were very accurate, to within $\pm 3\%$ of the exact solution. When the wave length was reduced, the errors tended to increase, which was to be expected, as the element mesh became too coarse. It was found that for linear elements such as those used here, approximately 8 elements per wave length should be used.

A further test was run to compare the present results with Mei's[1], using a finite element mesh and the Hankel function formulation, i.e. the fundamental solution. The results are for an incident wave with wave length $\lambda = 2\pi$ and unit incident surface elevation



...2 Interface

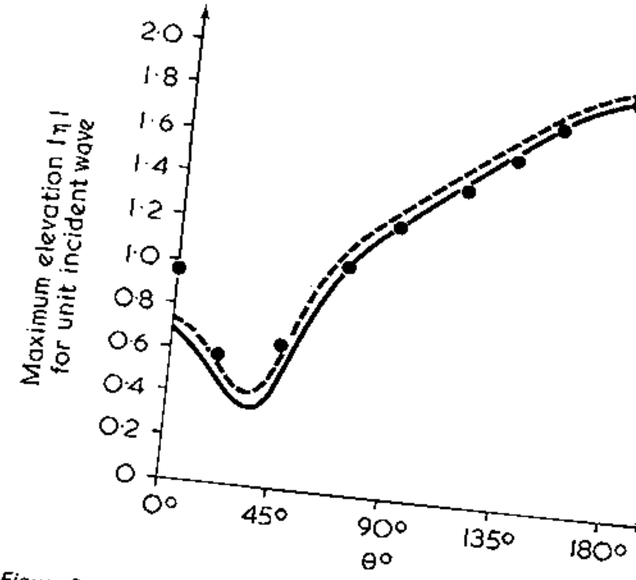


Figure 3 Results for maximum surface elevations round a circular cylinder. Wavelength, $2\pi M$; cylinder radius, 2M. (- - -) from Mei[1]; (—), analytical; (●), authors' results, coarse mesh with linear interpolation

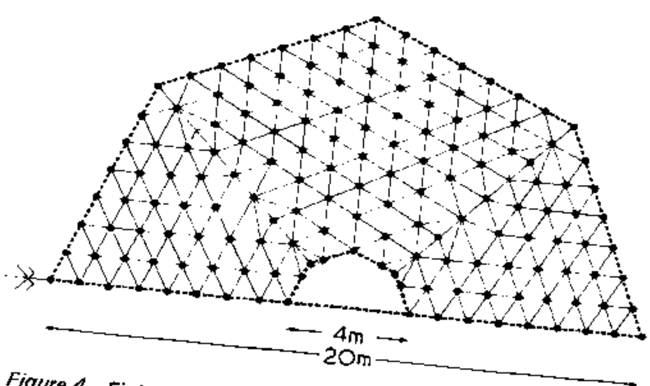


Figure 4 Finite element mesh

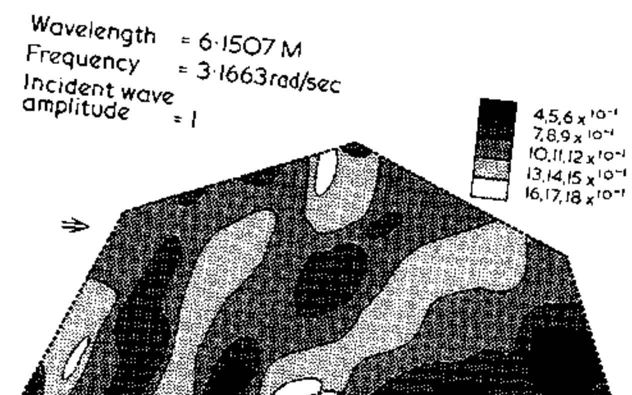


Figure 5 Surface elevations at time $t = 0$

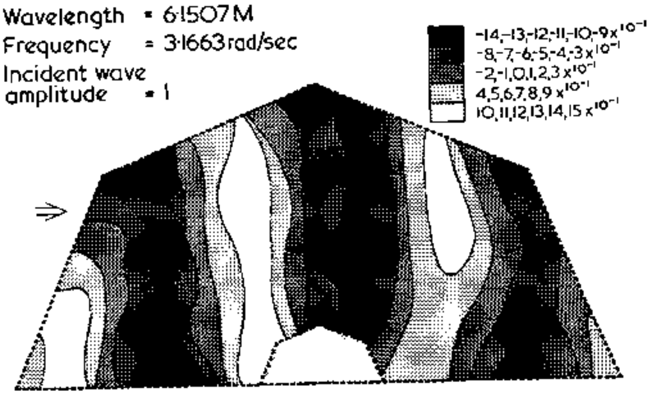


Figure 6 Maximum surface elevations.

for frequency $\omega = 3.1321$. Results of the exact solution, the finite element solution due to Mei and the solution obtained by the present authors are compared in *Figure 3*. The authors' solution compares favourably, taking into consideration the coarseness of the mesh around the cylinder. Mei's solution, for instance, uses 18 elements round the cylinder and better represents the geometry of the obstruction.

Conclusions

A simple method of formulating boundary elements for problems with radiation has been presented. It is simple to combine elements obtained in this way with finite elements and they have the advantage that they do not give full matrices. The basic idea can be applied to a wide range of problems, not only in hydraulics, but also in soil mechanics and general solid dynamics.

The technique has been applied to the two-dimensional Helmholtz equation but is even more advantageous when applied to three-dimensional problems.

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