

APPROXIMATE FUNDAMENTAL SOLUTIONS AND ALTERNATIVE FORMULATIONS

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ABSTRACT

The two main problem areas associated with the application of the Boundary Element Method to the solution of physical problems are:-

1. The determination of the form of the appropriate fundamental solution.
2. The evaluation of the integral of this singular function over the elements needed for the determination of the diagonal terms in the influence matrices.

In this paper the method of 'reference points' is described which avoids the second difficulty and a method for the construction of the fundamental solution for any linear partial differential equation is given. These methods are discussed with examples from two-dimensional potential and diffraction theory in order to demonstrate simply the principles involved. All the examples were solved using an Apple II microcomputer, demonstrating the efficiency and usefulness of the Boundary Element Method.

1 INTRODUCTION

It is not proposed in this paper to present an exhaustive description of the theory of fundamental solutions, the interested reader is referred to the theoretical expositions to be found in [1-6]. In this paper we shall be concerned with applications, particularly related to potential and diffraction problems. These examples were chosen because of their familiarity and in order to bring out the essential points in the simplest possible way. It is however necessary to clarify a number of points relating to the use of fundamental solutions in the Boundary Element Method at this stage.

The Fundamental Solution

Consider a medium in which we have a physical property represented by the function u which may be a function of space and time. If we know the governing equation for u and this equation is linear we can write this equation as

$$\mathcal{L}(u) = 0 \tag{1}$$

throughout the medium which we shall for the moment consider to be infinite in extent.

Here \mathcal{L} is a linear differential operator. The eigenfunctions ϕ_k corresponding to \mathcal{L} are defined by

$$\mathcal{L}(\phi_k) = \lambda_k \phi_k \tag{2}$$

where λ_k is a constant called the eigenvalue corresponding to the eigenfunction ϕ_k .

The form of the eigenfunctions will depend on the coordinate system used. For rectangular Cartesian coordinates we can choose the eigenfunctions to be the harmonic functions $\cos(Kx)$, $\sin(Kx)$, $\cos(Ky)$, etc., or using complex notation $e^{\pm iKx}$. The eigenvalues λ_k are then just polynomials involving powers of 'K' and 'iK'.

For infinite regions K is a continuous parameter so we shall write ϕ_k more explicitly as $\phi(\underline{K}, \underline{x})$. These eigenfunctions are usually orthonormal in the sense that

$$\int_{\text{all space}} \phi(\underline{K}, \underline{x}) \hat{\phi}(\underline{K}', \underline{x}) d\underline{x} = \delta(\underline{K} - \underline{K}') \tag{3}$$

where the hat $\hat{}$ denotes complex conjugation, δ is the Dirac delta function which is zero when $\underline{K} \neq \underline{K}'$ but infinite at $\underline{K} = \underline{K}'$. To define ϕ uniquely we require that

$$\int_{-\infty}^{\infty} \phi(\underline{K}, \underline{x}) \delta(\underline{K} - \underline{K}') d\underline{K} = \phi(\underline{K}', \underline{x}) \tag{4}$$

which is the replacement property of the delta function with respect to our eigenfunctions. $\int_{-\infty}^{\infty}$ represents integration over all space.

From (4):

$$\int_{-\infty}^{\infty} \delta(\underline{K} - \underline{K}') d\underline{K} = 1 \tag{5}$$

and

$$\int_{-\infty}^{\infty} \phi(\underline{K}, \underline{x}) \hat{\phi}(\underline{K}, \underline{x}) d\underline{K} = \delta(\underline{x} - \underline{x}') \tag{6}$$

This expansion enables the fundamental solution to be determined.

The fundamental solution u^* for the linear differential operator \mathcal{L} may be written

$$\mathcal{L}(u^*(\underline{x}, \underline{\xi})) = \delta(\underline{x} - \underline{\xi}) \quad (7)$$

and is a function of two points, the source point $\underline{\xi}$ and the observation point \underline{x} . (Note \mathcal{L} may be a polynomial of differentials with respect to the components of $\underline{\xi}$ or \underline{x} .) This fundamental solution represents the value of the field at point \underline{x} due to a point source at $\underline{\xi}$ (figure 1).

Using the properties of the delta function and the normalisation properties of the eigenfunctions the fundamental solution may be written

$$u^*(\underline{x}, \underline{\xi}) = \int_{-\infty}^{\infty} \phi(\underline{k}, \underline{x}) \frac{\hat{\phi}(\underline{k}, \underline{\xi})}{\kappa(\underline{k})} d\underline{k} \quad (8)$$

The formula given in (8) is a generalised transform. In fact in section 3 we shall use the Cartesian eigenfunctions $e^{-i\underline{k} \cdot \underline{x}}$ and (8) will then be an equation involving Fourier transforms.

2 REFERENCE POINTS

One of the main numerical problems associated with the fundamental solution is the evaluation of the diagonal terms of the influence matrices. These diagonal terms often consist of the evaluation of the integral of the fundamental solution over the singularity. Typically integrals like

$$G_{ii} = \int_{\Gamma} u^*(\underline{x}, \underline{\xi}) ds \quad (9)$$

and

$$H_{ii} = \int_{\Gamma} \frac{\partial u^*(\underline{x}, \underline{\xi})}{\partial n} ds \quad (10)$$

are sought, where ds is an elemental length along the element considered.

These problems may be simply avoided. Consider the general second order linear partial differential equation:-

$$a'u_{xx} + 2b'u_{xy} + c'u_{yy} + d'u_x + e'u_y + f'u = g' \quad (11)$$

where the coefficients a' , b' , c' , d' , e' , f' and g' are functions of x and y only, and the subscripts x and y denote differentiation w.r.t. these variables.

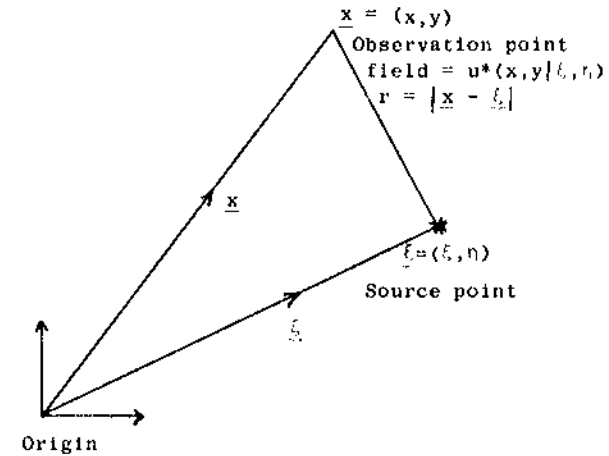


FIGURE 1 NOTATION FOR DEFINITION OF THE FUNDAMENTAL SOLUTION

If $b^2 - 4ac > 0$ then the equation is elliptic, we shall consider this case. Equation (11) may always be reduced to the canonical form (see reference 6, Chapter 3)

$$\mathcal{L}(u) = u_{xx} + u_{yy} + au_x + bu_y + cu \quad (12)$$

define the adjoint operator to \mathcal{L} , \mathcal{L}^* such that

$$\mathcal{L}^*(v) = v_{xx} + v_{yy} - av_x - bv_y + (c - a_x - b_y)v \quad (13)$$

Then with n as the outward normal, for u and v any sufficiently differentiable functions:-

$$\int_{\Omega} v \mathcal{L}(u) - u \mathcal{L}^*(v) dV = \int_{\Gamma} \left[v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} + (a \frac{\partial x}{\partial n} + b \frac{\partial y}{\partial n}) uv \right] dS \quad (14)$$

Ω is the problem region and Γ is the boundary.

Then if we choose v to be the fundamental solution $u^*(\underline{x}, \underline{\xi})$ defined by:-

$$\mathcal{L}^*(u^*(\underline{x}, \underline{\xi})) = \delta(\underline{x} - \underline{\xi}) \quad (15)$$

then we may use the selective property of the delta function to reduce (14) to:-

$$cu(\underline{x}) = \int_{\Gamma} \left[u^* \frac{\partial u}{\partial n} - u \frac{\partial u^*}{\partial n} \right] + \left[a \frac{\partial x}{\partial n} + b \frac{\partial y}{\partial n} \right] uu^* dS(\underline{\xi}) \quad (16)$$

we have taken the integrations to be over the variable $\underline{\xi}$

where for smooth boundaries

$$\begin{aligned} c &= 0 & \underline{x} &\notin \Omega \\ c &= \frac{1}{2} & \underline{x} &\in \Gamma \\ c &= 1 & \underline{x} &\in \Omega \end{aligned}$$

Notice that u^* is a function of two variables and that (16) is an identity for all values of the \underline{x} reference point.

Consider the above with $a = b = 0$ then

$$cu(\underline{x}) = \int_{\Gamma} u^*(\underline{x}, \underline{\xi}) \left[\frac{\partial u(\underline{\xi})}{\partial n} - u(\underline{\xi}) \frac{\partial u^*(\underline{x}, \underline{\xi})}{\partial n} \right] dS(\underline{\xi}) \quad (17)$$

Then if we choose \underline{x} to be a reference point \underline{x}_i outside Ω , say

$$\underline{x} = \underline{x}_i$$

then $c = 0$ and (17) may be written:-

$$\int_{\Gamma} u^*(\underline{x}_i, \underline{\xi}) \frac{\partial u(\underline{\xi})}{\partial n} dS(\underline{\xi}) = \int_{\Gamma} u(\underline{\xi}) \frac{\partial u^*(\underline{x}_i, \underline{\xi})}{\partial n} dS(\underline{\xi}) \quad (18)$$

or approximately for n elements Γ_j .

$$\begin{aligned} \sum_j \frac{\partial u(\underline{\xi}_j)}{\partial n} \int_{\Gamma_j} u^*(\underline{x}_i, \underline{\xi}) dS(\underline{\xi}) \\ = \sum_j u(\underline{\xi}_j) \int_{\Gamma_j} \frac{\partial}{\partial n} u^*(\underline{x}_i, \underline{\xi}) dS(\underline{\xi}) \end{aligned} \quad (19)$$

If we discretise Γ into boundary elements then we can now choose n such reference points. We have now constructed n equations for the n unknown u values on the boundary.

WE NOW HAVE NO SINGULAR INTEGRALS TO EVALUATE. One disadvantage of this method is that the influence matrices become less diagonally dominant, with the resulting numerical problems, as the diagonal terms now correspond to an evaluation of u^* between the i th element and i th reference point. This difficulty may be overcome by choosing the reference point close to the corresponding element.

Example 2-D Laplace equation

For the two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0 \quad (20)$$

$$\nabla^2 u^*(\underline{\xi}, \underline{x}_i) = \delta(\underline{\xi} - \underline{x}_i) \quad (21)$$

and

$$u^*(\underline{\xi}, \underline{x}_i) = \frac{1}{2\pi} \ln r_{ij} \quad (22)$$

where

$$r_{ij} = |\underline{\xi}_j - \underline{x}_i| \quad (23)$$

then (14) becomes

$$\sum_j \frac{\partial u(\underline{\xi}_j)}{\partial n} \int_{\Gamma_j} \ln r_{ij} dS = \sum_j u(\underline{\xi}_j) \int_{\Gamma_j} \frac{\partial}{\partial n} \ln r_{ij} dS \quad (24)$$

note that at no point on the boundary element Γ_j does r_{ij} become zero not even when $i = j$. In the example the reference points \underline{x}_i were chosen to be on the normal to element i outside the problem region (Figure 2) distance \bar{r} from the element where \bar{r} is the length of the element.

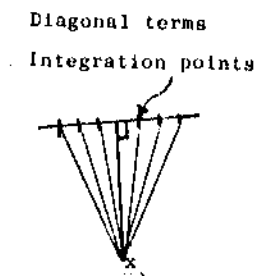
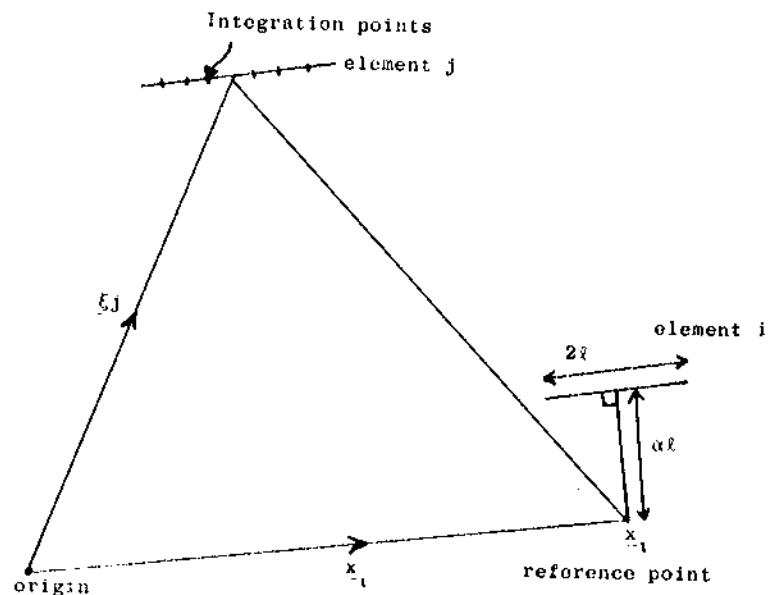


FIGURE 2 THE USE OF REFERENCE POINTS IN BOUNDARY ELEMENTS

n equations like (24) may be written down, one corresponding to each reference point position \underline{x}_i .

The example considered is shown in Figure 3 and follows the geometry and discretisations given in [7]. As in [7] the example was solved on a desktop microcomputer (the Apple II).

The problem was solved for a number of values of the scale parameter α , which determines the distance of the reference points from their corresponding elements. $\alpha = 0$ corresponds to the conventional Boundary Element Formulation. The results are presented element by element in Figure 4. The value of α giving the best results was about 1.

For more complicated equations the evaluation of the diagonal terms will often involve the use of complicated special functions. For the two dimensional diffraction case the fundamental solution is given by

$$\nabla^2 u^*(\underline{x}, \underline{\xi}) + k^2 u^*(\underline{x}, \underline{\xi}) = \delta(\underline{x} - \underline{\xi}) \quad (25)$$

i.e.

$$u^*(\underline{x}, \underline{\xi}) = -\frac{1}{4i} H_0^{(2)}(Kr) \quad (26)$$

where $r = |\underline{x} - \underline{\xi}|$

and $H_0^{(2)}$ is a Hankel function of order zero of the second kind.

The diagonal term is of the form

$$G_{ii} = -\frac{1}{4i} \int_{-L}^L H_0^{(2)}(Kr) dS \quad (27)$$

An integral which when evaluated involves Struve and Bessel functions. The use of reference points avoids the need for the use of Struve functions altogether.

Example

Consider the diffraction problem in Figure 5. A wave of the form $e^{i(kx - \omega t)}$ is incident on a circular cylinder from the right.

The zero normal flux on the cylinder determines the boundary condition on the cylinder to be:-

$$\frac{\partial u}{\partial n} = -\frac{\partial u_I}{\partial n} \quad (28)$$

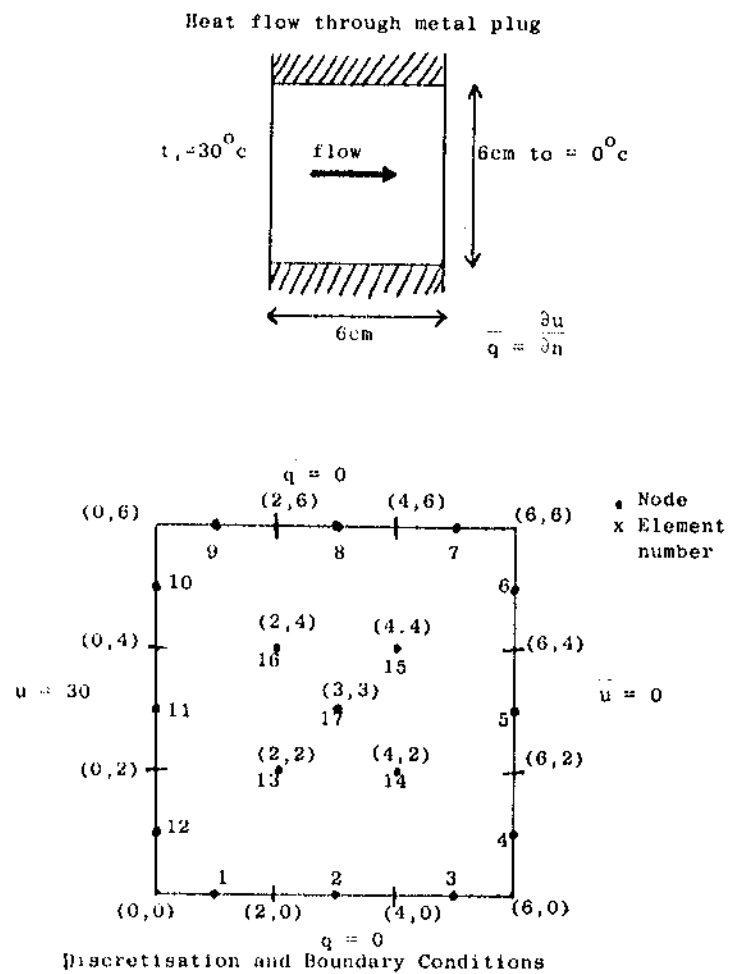


FIGURE 3 METAL PLUG EXAMPLE

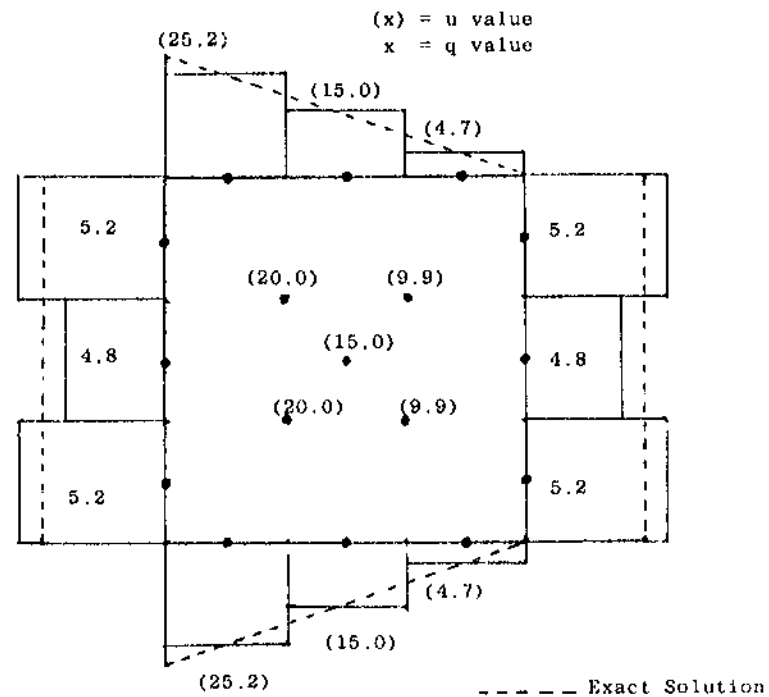


FIGURE 4 METAL PLUG EXAMPLE RESULTS

REFERENCE POINTS - COMPARISON OF RESULTS
SCALE FACTOR

ELEMENT	0.†	0.5	1.5	3
1	(25.2)	(25.5)	(25.8)	(25.9)
2	(15.0)	(14.9)	(14.99)	(15.0)
3	(4.77)	(4.35)	(4.22)	(4.11)
4	-5.30	-4.97	-5.09	-4.92
5	-4.88	-4.75	-4.97	-5.21
6	-5.30	-4.97	-5.09	-4.92
7	(4.77)	(4.35)	(4.22)	(4.11)
8	(15.0)	(14.9)	(14.99)	(15.0)
9	(25.2)	(25.5)	(25.8)	(25.9)
10	5.30	5.16	5.09	4.92
11	4.87	4.80	4.97	5.21
12	5.30	5.16	5.09	4.92
13	(20.0)	(19.8)	(20.0)	(20.0)
14	(9.97)	(9.82)	(9.96)	(9.96)
15	(9.97)	(9.82)	(9.96)	(9.96)
16	(20.0)	(19.8)	(20.0)	(20.0)
17	(15.0)	(14.8)	(14.99)	(14.99)

† = 0 corresponds to the classical boundary element method
 (x) u value
 x q value

TABLE 1

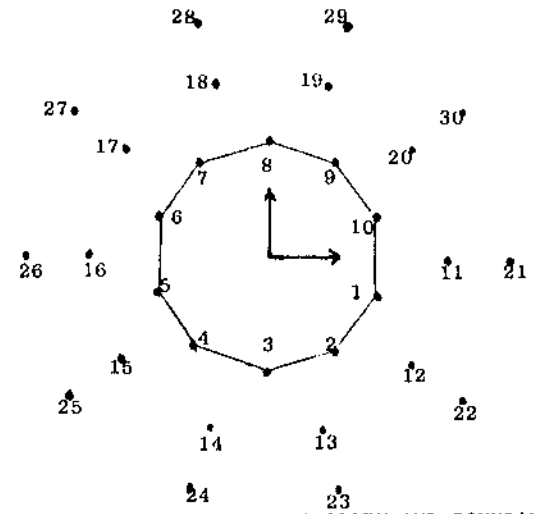
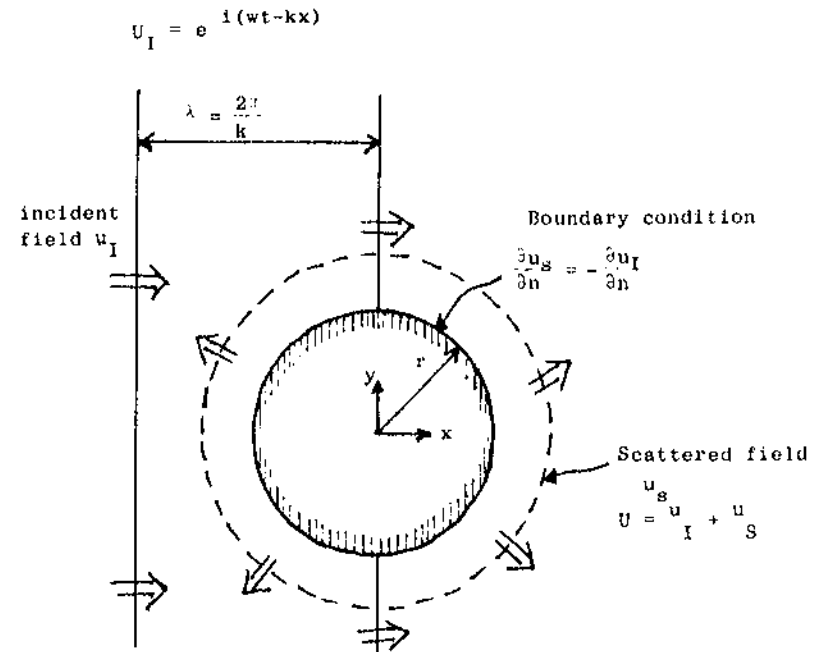


FIGURE 5 DIFFRACTION PROBLEM AND BOUNDARY ELEMENT DISCRETISATION

where $u = U_S + U_I$

and U_I is the incident field

U_S is the scattered field due to the presence of the cylinder

The results for this problem for radius = 1 and wavelength = 2 are given in Table 2. The values given are the complex moduli of the scattered field representing the amplitude of the oscillations of this field. This field is calculated using the reference point method and the classical Boundary Element Method for comparison.

In the examples considered above it was not strictly necessary to use reference points. In some cases, however, the integrals necessary for evaluation of the diagonal terms are very cumbersome or not known in closed form. This eventuality usually necessitates the use of an elaborate and slow numerical integration scheme.

3 NUMERICAL AND APPROXIMATE FUNDAMENTAL SOLUTIONS

As an example of the methods involved in the use of numerical and approximate fundamental solutions consider the two-dimensional reduced Helmholtz equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \quad (29)$$

using (8) and the normalised eigenfunctions given by

$$\phi(K_1, K_2; x, y) = \frac{1}{\sqrt{2}} \exp(ik_1 x + ik_2 y) \quad (30)$$

then the fundamental solution may be written

$$u^*(x, y | \xi, \eta) = \frac{1}{(2\pi)^2} \int \int \frac{e^{ik_1(\xi-x)} e^{ik_2(\eta-y)}}{(k^2 - K_1^2 - K_2^2)} dk_1 dk_2 \quad (31)$$

In this case the integral may be evaluated using contour integration or coordinate transformations to give

$$u^*(x, y | \xi, \eta) = -\frac{1}{4i} H_0^{(2)}(K|x - \xi|) \quad (32)$$

For more complicated linear differential operators (31) becomes

DIFFRACTION PROBLEM MODULI OF SCATTERED POTENTIAL, COMPARISON OF RESULTS

ELEMENT OR NODE	SCALE FACTOR α			APPROXIMATE FUNDAMENTAL SOLUTION
	0	0.5	1.0	
1	1.23	1.29	1.24	1.28
2	0.516	0.520	0.507	0.546
3	0.482	0.506	0.455	0.480
4	0.815	0.803	0.776	0.808
5	0.891	0.894	0.924	0.917
11	1.15	1.18	1.15	1.17
12	0.326	0.358	0.340	0.346
13	0.134	0.152	0.135	0.145
14	0.446	0.437	0.429	0.432
15	0.630	0.616	0.637	0.613
21	0.975	1.00	0.976	0.996
22	0.308	0.335	0.318	0.317
23	0.137	0.153	0.136	0.146
24	0.431	0.421	0.416	0.418
25	0.516	0.504	0.524	0.504

Wavelength = 2

Radius of cylinder = 1

The omitted nodes have symmetric values 5+10 = 1+5 etc

N.B. Only ten boundary elements were used for a wavelength of only twice the radius

TABLE 2

$$u^*(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{ik_1(-x)} e^{ik_2(z-y)}}{P(ik_1, ik_2)} dk_1 dk_2 \quad (33)$$

where P is a polynomial of ik_1 and ik_2 and the following correspondences may be made between the components of the operator and the polynomial

$$\frac{\partial}{\partial x} \leftrightarrow ik_1, \text{ and } \frac{\partial}{\partial y} \leftrightarrow ik_2 \quad (34)$$

The integral (33) may be evaluated numerically for all relevant points (x,y) and (z,y) used in the problem geometry, although the singularities corresponding to the real roots of P (k₁, k₂) need some considerable care. Integrals of u* over elements of the form (9) may be evaluated analytically with respect to dS by change of the order of integrations.

It now only remains to consider the evaluation of integrals of type (10) of the normal derivative of u*. Remembering that

$$\frac{\partial u^*}{\partial n} = \frac{\partial u^*}{\partial z} \frac{\partial z}{\partial n} + \frac{\partial u^*}{\partial y} \frac{\partial y}{\partial n} \quad (35)$$

then $\frac{\partial u^*}{\partial z}$ and $\frac{\partial u^*}{\partial y}$ may be evaluated by differentiation of (33) with respect to z and y underneath the integral signs. $\frac{\partial z}{\partial n}$ and $\frac{\partial y}{\partial n}$ are for straight elements just dependent on the element orientation. The integrals over dS along the element may be eliminated as before. In fact

$$\int \frac{\partial u^*}{\partial n} dS = \int \frac{\sin \theta}{\sin \theta} \frac{\partial u^*}{\partial z} - \int \frac{\cos \theta}{\cos \theta} \frac{\partial u^*}{\partial y} dS \quad (36)$$

where θ is the angle of the element makes with the horizontal. (For $\theta = 0$ or π (36) is even simpler.)

The wholly numerical approach described above may not be appropriate in all cases. In fact in the example considered above an approximate fundamental solution may be used. The justification of this is that the most singular terms of the fundamental solution arise from the second derivative terms in the corresponding linear operator. In fact these are the terms which correspond to the Laplace equation which has a simple and well known fundamental solution.

Consider the limits for large and small arguments of the diffraction fundamental solution given in (26). We have:-

$$\begin{array}{l} \text{Small } Kr \\ -\frac{1}{2\pi} \log Kr \sim -\frac{1}{4i} H_0^{(2)}(Kr) \sim \frac{1}{4i} \sqrt{\frac{2}{\pi Kr}} e^{-i(Kr - \frac{\pi}{4})} \end{array} \quad \begin{array}{l} \text{Large } Kr \\ (Kr) \sim \frac{1}{4i} \sqrt{\frac{2}{\pi Kr}} e^{-i(Kr - \frac{\pi}{4})} \end{array} \quad (37)$$

For fixed K and small Kr the diffraction fundamental solution behaves like the potential fundamental solution. For correct choice of element size the integral

$$G_{ij} = \int_{-l}^l u^* dS$$

for diffraction problems may be evaluated using the potential fundamental solution in place of u*. For widely spaced elements (with respect to the wave length considered) the approximation corresponding to Large Kr for u* may be used for the G_{ij} and H_{ij} terms.

Consider the diffraction problem in Figure 5. This time the small Kr approximation has been used in the evaluation of the diagonal terms, this is a valid approximation for elements of suitable size. The last column in Table 2 shows the small effect of this approximation on the results.

4 CONCLUSIONS

It has been shown that it is usually possible to avoid the need to integrate the fundamental solution across the singularity which occurs when the observation point and source point coincide, by the use of reference points.

In section 3 a method is outlined for solving second order partial differential equations when the explicit form of the fundamental solution is not known or very cumbersome, by numerical evaluation of specific integral representations of the solution.

Where numerical integration is not appropriate it is often possible to use an approximate fundamental solution, consequently increasing speed and decreasing complexity. This is a consequence of the fact that the singular part of most fundamental solutions is a variant of the potential fundamental solution with the same number of dimensions as the problem considered.

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AN EFFICIENT ALGORITHM FOR THE NUMERICAL EVALUATION OF
BOUNDARY INTEGRAL EQUATIONS

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ABSTRACT

The usual approach to the solution of boundary integral equations is to represent the unknown vector by a piecewise constant, linear, or quadratic function over the given mesh subdivision. These representations have the advantages of consistency, ability to integrate the equations for the given functional approximations, and, in general, improved accuracy as the degree of approximation is increased. While adequate for many problems, special requirements arise for certain nonlinear problems, e.g., plasticity, where the integral equations must be solved for each load increment. In the present paper a special numerical algorithm is outlined in which the unknown vector is represented as a combination of a Fourier series and piecewise linear function. The piecewise linear function is used only in high gradient regions of the unknown vector thus permitting an excellent representation with relatively few Fourier terms. The algorithm is compared with a linear representation alone for two problems which show the effects of multiple connectivity, sharp corners and discontinuous loading. For comparable accuracy both problems show a significant improvement in computer time required.

INTRODUCTION

The boundary integral method [3,5,7] is fast becoming a generally accepted alternative to finite element, finite difference methods at least for linear problems. The principal reason is that the problem dimension is reduced by one, e.g., a two-dimensional problem is reduced to a line integral equation on the boundary. After discretizing, the integral equation formulation results in a lower order set of algebraic equations which implies less computer time. There are also certain inherent accuracy improvements, particularly for interior information.