

Boundary elements in fluid-structure interaction problems rotational shells

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The use of the finite element method for structural analysis is now commonplace. In the analysis of offshore structures parts of the structure are often in contact with a fluid either internally or externally. The fluid will affect the natural frequencies and mode shapes of the structure. In this paper a general purpose package of computer programs based on the boundary element method, for the solution of such problems is described. The specific example of the forced response of a pressure vessel containing a compressible fluid with a free surface is discussed as a worst case example. Simpler examples of situations incorporating only some of the features of this problem are also discussed.

Introduction

There have been a number of attempts to analyse fluid structure interaction problems using various techniques involving a mixture of finite element, boundary element and analytic representations for the fluid and structure, often the structure may be assumed to be rigid and inelastic, as in the case of diffraction problems.¹

The first attempts at solving the fluid/structure interaction problems represented the fluid forcing on the structure analytically.^{2,3} The motion of the structure, however, was not assumed to change the nature of the forcing appreciably although some attempts have been made to generalize the Morison equation to take this effect into account.^{4,5} The resulting approach has often involved a cyclic iterative procedure.

The next stage was to consider the fluid as three-dimensional finite elements joined at the structural nodes to the structure's finite element mesh. For the external problems this, however, involves the use of some kind of asymptotic matching procedure (as in references 6 and 7) or the use of, so-called, infinite finite elements. Another alternative is to use the Sommerfeld radiation condition on the boundary of the finite element mesh.⁸ The use of finite elements for the fluid, and some kind of matching on the boundary of the finite element mesh has perhaps been superseded for most problems by the implementation of the boundary method⁹ which has three important advantages.

(1) The representation of the fluid is effected using two-dimensional surface boundary elements which can be directly identified with the finite elements on the surface of contact with the structure.

(2) With careful choice of the fundamental solution infinite fluid regions may be represented in the same way.

(3) For the internal problem (e.g. pressure vessels) a free surface boundary condition may be used to eliminate all nodal unknowns on the liquid surface not in contact with the structure.

Two methods of formally joining finite and boundary element regions are given in Brebbia and Walker,¹⁰ the problem is further discussed by Shaw.¹¹ Suitable fundamental solutions for infinite and semi-infinite regions are given in reference 9.

These methods have the added advantage that no iterative procedure is involved in determining the back reaction of the structure on the fluid.

Definition of the problem

The problem to be solved is to determine the forced response of the vessel shown in *Figure 1*.

The compressible liquid comes up to the level indicated and is agitated internally by two Rushton impellers which rotate at a fixed speed. The convective velocity of the liquid is well subsonic so the acoustic field is essentially decoupled from the convective field which gives rise to a regular and (assumed known) forcing on the structure.

Because the internal dimensions of the vessel correspond to about one wavelength at the frequency of agitation, we are near acoustic resonance and the acoustic field is the main mechanism of energy transfer through the field.

The procedure for the solution of fluid/structure interaction problems is shown in *Figure 2*. *Figure 1* gives some idea of the geometry of the problem considered. A coarser

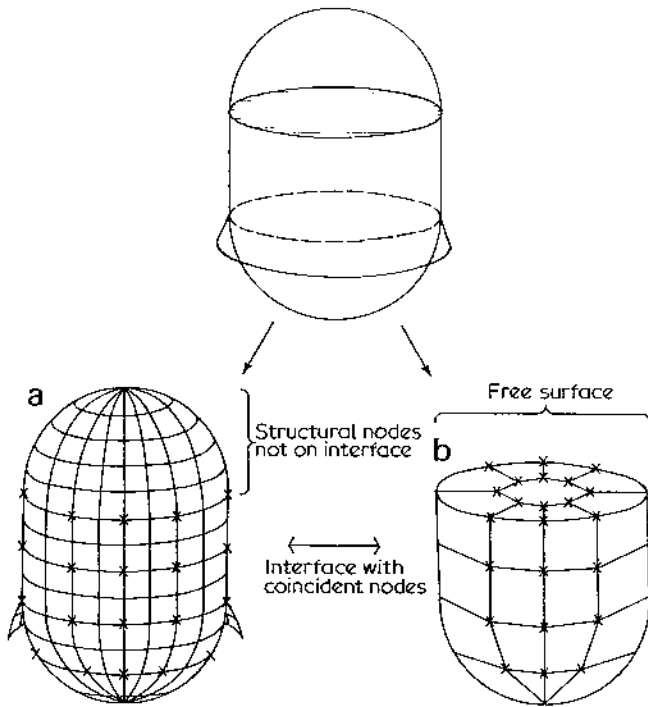


Figure 1 Idealization of fluid/structure system for pressure vessel. (a), Finite element mesh; (b) Boundary element mesh.

boundary element mesh was chosen for representation of the fluid than the finite element mesh, as the determining factor for the choice for the mesh is the ratio of the acoustic wavelength to the element side lengths. There were some complex internal structural elements which necessitated a very detailed finite element representation.

Boundary element formulation (fluid region)

In this section we look at the application of the BEM to the three-dimensional acoustic field problem. For the acoustic field within the fluid we may define a velocity potential ϕ by:

$$u = \nabla\phi = \text{Grad } \phi \tag{1}$$

where u is the (Eulerian) velocity of the fluid. The governing equation of the fluid may now be written:

$$\nabla^2\phi = \frac{1}{c^2} \frac{\partial^2\phi}{\partial t^2} \tag{2}$$

Assuming a harmonic time dependence for ϕ of the form $e^{i\omega t}$, equation (2) becomes:

$$\nabla^2\phi + \kappa^2\phi = 0 \tag{3}$$

where:

$$\kappa = \frac{\omega}{c} \tag{4}$$

Or more generally, if we have a distribution $m(r)$ of sources (with r the position vector) we may write:

$$\nabla^2\phi + \kappa^2\phi = m(r) \tag{5}$$

consider:

$$\nabla^2G + \kappa^2G = \delta(r - r') \tag{6}$$

where G is the free space Green's function or fundamental solution, which is a function of two variables the source

point r' , and the observation point r ; δ is the Dirac delta function.

The solution of equation (6) is given explicitly by:

$$G(r, r') = \frac{-e^{-i\kappa r}}{4\pi r} \tag{7}$$

where $r = |r - r'|$

Green's theorem states, that for any two functions ϕ and ψ which are sufficiently differentiable for ∇^2 to exist we may write:

$$\int_V \{\psi \nabla^2\phi - \phi \nabla^2\psi\} dV = \int_S \left\{ \psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n} \right\} dS \tag{8}$$

where the regions of integration are indicated in Figure 3 and n is the normal to V .

If we choose ψ to be the fundamental solution G , then using (6) and the selective property of the delta function we can eliminate the volume integral to obtain:

$$\phi(r') = - \int_S \left\{ G(r, r') \frac{\partial\phi}{\partial n} - \phi \frac{\partial G(r, r')}{\partial n} \right\} dS(r) \tag{9}$$

This is an identity for all r' in V (the Helmholtz formula).

In our case we have no net creation or destruction of fluid in our problem region and we have put $m(r) \equiv 0$. If

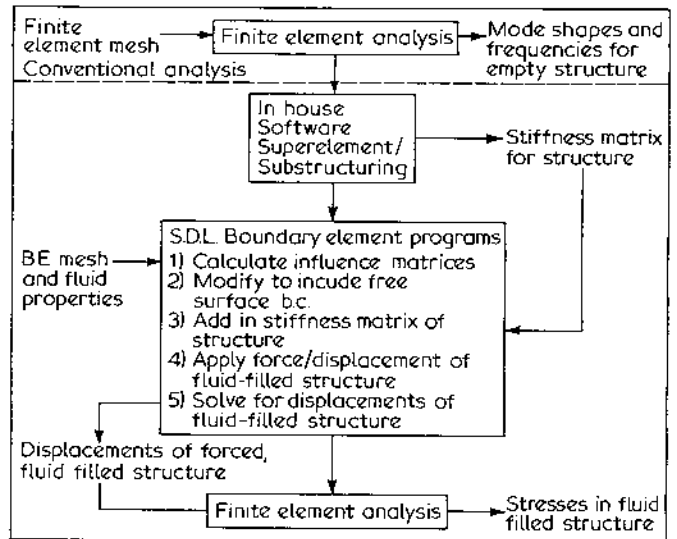


Figure 2 Solution procedure fluid/structure interaction

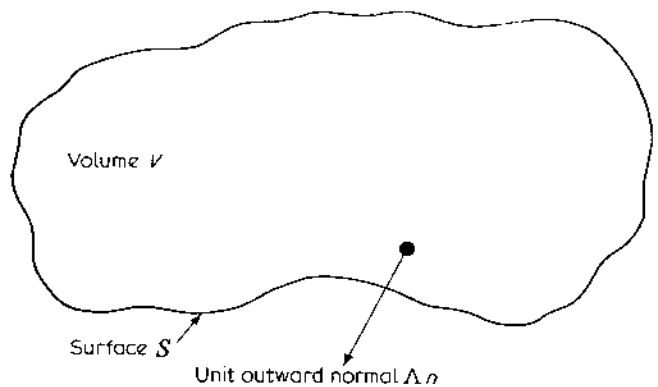


Figure 3 Problem region for Green's theorem

we now choose r' to be a point on the boundary S , then (9) becomes:

$$c\phi(r') = - \int_S G(r', r) \frac{\partial \phi}{\partial n} - \phi \frac{\partial G(r', r)}{\partial n} dS(r) \quad (10)$$

where:

$$c = \frac{\alpha}{4\pi} \quad (11)$$

and α is the solid angle interior to Γ at r' (usually 2π for a smooth surface).

We now discretize Γ into N flat 'facets' or 'elements' of area $A^{(j)}$, then $c = 1/2$ (Figure 1).

We choose to use constant elements to avoid the corner problems here, we approximate ϕ over element j by its value at the centroid of the area $A^{(j)}$ say. Similarly:

$$\frac{\partial \phi}{\partial n} = \left[\frac{\partial \phi}{\partial n} \right]_j = q_j \quad (12)$$

Equation (10) may then be written:

$$\frac{\phi_i}{2} = - \sum_{j=1}^N \left\{ \int_{S_j} G \frac{\partial \phi}{\partial n} dS_j - \int_{S_j} \phi \frac{\partial G}{\partial n} dS_j \right\} \quad (13)$$

hence approximately:

$$\frac{\phi_i}{2} = - \sum_{j=1}^N q_j \int G dS_j + \sum_{j=1}^N \phi_j \int \frac{\partial G}{\partial n} dS_j \quad (14)$$

making the following changes in notation:

Replace G by g_{ij} where $r_j = r, r_i = r'$, then:

$$g_{ij} = \frac{-\exp\{-ik|r_j - r_i|\}}{4\pi|r_j - r_i|} \quad (15)$$

$$h_{ij} = \frac{\partial g_{ij}}{\partial n} \quad (16)$$

Note that g_{ij} is symmetric in i and j . Let:

$$G_{ij} = \int_{S_j} g_{ij} dS_j \quad (17)$$

$$\hat{H}_{ij} = \int_{S_j} h_{ij} dS_j \quad (18)$$

It should be noted that because we are using the constant element formulation and not the source approximation (approximating G_{ij} by g_{ij} at the midpoint) we shall not get symmetric matrices for either G or H ; even if we choose our variables to be scaled on the element areas. This is because of the finite size of the elements. Equation (14) may be written:

$$\frac{\phi_i}{2} = \sum_{j=1}^N \{ \hat{H}_{ij} \phi_j - G_{ij} q_j \} \quad (19)$$

Allowing:

$$H_{ij} = \hat{H}_{ij} - \frac{1}{2} \delta_{ij} \quad (20)$$

where δ_{ij} is the Kronecker delta defined by:

$$\begin{aligned} \delta_{ij} &= 0 & i \neq j \\ \delta_{ij} &= 1 & i = j \end{aligned} \quad (21)$$

equation (19) may be more concisely written:

$$\sum_{j=1}^N H_{ij} \phi_j = \sum_{j=1}^N G_{ij} q_j \quad (22)$$

Transformation of variables to finite element form

As we are considering harmonic time dependence we may write the normal velocity through element j in terms of the normal displacement as:

$$q_j = i\omega U_j \quad (23)$$

and using Bernoulli's equation we can write the velocity potential on element j in terms of the pressure or normal force F_j on that element.

$$\phi_j = - \frac{P_j}{i\omega\rho} = - \frac{F_j}{A^{(j)}i\omega\rho} \quad (24)$$

where $A^{(j)}$ is the area of the j th element.

Hence equation (22) becomes:

$$\sum_j \left[\frac{H_{ij}}{A^{(j)}} \right] F_j = \omega^2 \rho \sum_j G_{ij} U_j \quad (25)$$

Or

$$HF = \omega^2 \rho GU \quad (26)$$

hence:

$$F = \{\omega^2 \rho H^{-1} G\} U \quad (27)$$

or:

$$F = AU \quad (28)$$

where:

$$A = H^{-1}G \quad (29)$$

Equation (28) is of the same form as the finite element 'stiffness' formulation linking forces and displacements.

The integrals G_{ij} and H_{ij} were evaluated numerically using four point Gaussian integration, i.e. 16 points were used on each element.¹² The following two points should be noted:

(1) $H_{ij} = 0$ as it represents the integral of the flux normal to the element from a point source in the plane of the element. This singular term need not therefore be evaluated explicitly.

(2) In the incompressible limit, i.e. $c \rightarrow \infty$ and $\kappa \rightarrow 0$ we have the usual potential theory formulation, H and G being independent of frequency. The matrix A then has the form of a mass or 'added mass' matrix as we would expect. For convenience, in this paper the term stiffness matrix will be assumed to incorporate the mass matrix in the form $K - \omega^2 M$ where it exists.

Application of the free surface boundary condition

On the free surface of a compressible fluid we have:

$$\frac{\partial \phi}{\partial z} = \frac{\omega^2 \phi}{g} \quad (30)$$

for harmonic time dependence, which becomes after discretization:

$$F_j = g\rho A^{(j)} U_j \quad (31)$$

We denote free surface values of the variables by a subscript o then:

$$F_{oj} = g\rho A^{(j)}U_{oj} \tag{32}$$

From above:

$$F = AU \tag{33}$$

or in partitioned form:

$$\begin{bmatrix} F \\ F_o \end{bmatrix} = \begin{bmatrix} A_{11} & A_{1o} \\ A_{o1} & A_{oo} \end{bmatrix} \begin{bmatrix} U \\ U_o \end{bmatrix} \tag{34}$$

Therefore:

$$F_o = A_{o1}U + A_{oo}U_o = \rho g \{ A^{(j)}U_{oj} \} \tag{35}$$

therefore:

$$A_{o1}U = [-A_{oo} + \rho g A^{(j)}I] U_o \tag{36}$$

Or:

$$U_o = [-A_{oo} + \rho g A^{(j)}I]^{-1} A_{o1}U$$

but:

$$\begin{aligned} F &= A_{11}U + A_{1o}U_o \\ &= [A_{11} + A_{1o}(-A_{oo} + \rho g A^{(j)}I)^{-1} A_{o1}] U \end{aligned} \tag{37}$$

We may write this as:

$$F = A'U \tag{38}$$

We have now expressed the forces at nodes on the fluid/structure interface in terms of the displacements at these nodes only. A' is the modified matrix after the application of the free surface boundary condition (31).

Joining of finite and boundary element regions

When considering the joining of finite and boundary element regions one fundamental problem becomes apparent.

The boundary element formulation gives results in terms of normal displacements and forces, whereas the finite element scheme involves displacements and forces in the three global coordinate directions. The procedure chosen was to augment the boundary element matrix A' so that it now expresses the relationship between displacements and forces in the three coordinate directions.

Using the subscript n to emphasize the fact that all variables are in the normal direction we may write (38) as:

$$F_{ni} = \sum_j A_{ij} U_{nj} \tag{39}$$

If we represent an orthogonal pair of unit tangent vectors in the plane of element i by t_i and t'_i then forces and displacements may in general be written:

$$F_i = F_{ni} \cdot N_i + F_{ti} \cdot t_i + F'_{ti} \cdot t'_i \tag{40}$$

$$U_i = U_{ni} \cdot N_i + U_{ti} \cdot t_i + U'_{ti} \cdot t'_i \tag{41}$$

Looking at the x component we may write:

$$F_{xi} = \sum_j A_{ij} N_{xi} U_{nj} + \sum_j B_{ij} t_{xi} U_{tj} + \sum_j B'_{ij} t'_{xi} U'_{tj} \tag{42}$$

similarly for the other two components.

We know that as we are dealing with an inviscid fluid displacements tangential to the fluid boundary have no effect, hence:

$$B_{ij} \equiv 0 \tag{43}$$

and $B'_{ij} \equiv 0$ for all i and j .

Hence combining equations (41) and (42) we have:

$$\begin{aligned} F_{xi} &= \sum_j A_{ij} N_{xi} N_{xj} U_{xj} \\ &+ \sum_j A_{ij} N_{xi} N_{yj} U_{yj} \\ &+ \sum_j A_{ij} N_{xi} N_{zj} U_{zj} \end{aligned} \tag{44}$$

writing $A_{ij}^{xy} = A_{ij} N_{xi} N_{yj}$ etc. we have for the three components:

$$\begin{bmatrix} F_{xi} \\ F_{yi} \\ F_{zi} \end{bmatrix} = \begin{bmatrix} A_{ij}^{xx} & A_{ij}^{xy} & A_{ij}^{xz} \\ A_{ij}^{yx} & A_{ij}^{yy} & A_{ij}^{yz} \\ A_{ij}^{zx} & A_{ij}^{zy} & A_{ij}^{zz} \end{bmatrix} \begin{bmatrix} U_{xj} \\ U_{yj} \\ U_{zj} \end{bmatrix} \tag{45}$$

The equations are now in the correct form for combination.

Consider the discretized problem depicted in *Figure 4* divided into two regions Ω^2, Ω^1 . Ω^1 is the fluid region.

The two regions are bounded by surfaces Γ^1 and Γ^2 as follows:

- (1) Γ^1_1 that part of the surface of Ω^1 not in contact with the structure (the free surface).
- (2) Γ^1_2 that part of Ω^1 in contact with the structure (the interface nodes).
- (3) Γ^2_1 that part of Ω^2 in contact with the fluid.
- (4) Γ^2_2 that part of Ω^2 not in contact with the fluid.

In *Figure 4* the nodes and elements of the interfaces Γ^1_1 and Γ^2_1 are shown as two distinct surfaces for clarity. The nodal unknowns F and U are in fact related by two conditions on the interface.

- (1) Compatibility, the nodal displacements at corresponding nodes are equal on the interface:

$$U^1_I = U^2_I = U_I \tag{46}$$

- (2) Equilibrium, the interfacial forces; fluid on structure, structure on fluid, must be equal and opposite:

$$F^1_I = -F^2_I = F_I \tag{47}$$

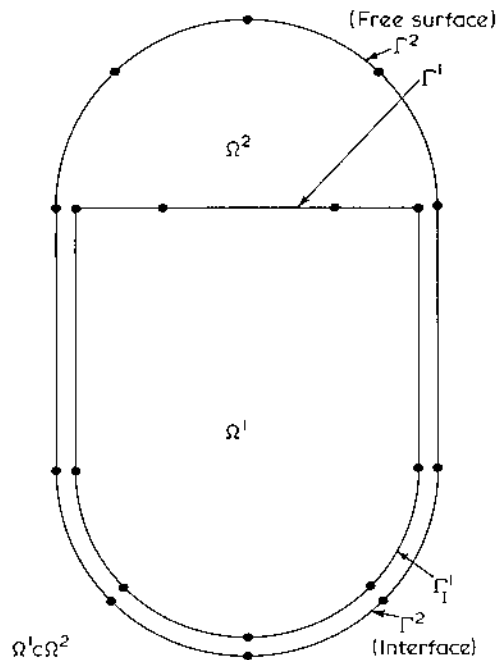


Figure 4 Section through problem region showing subsets of bounding surfaces

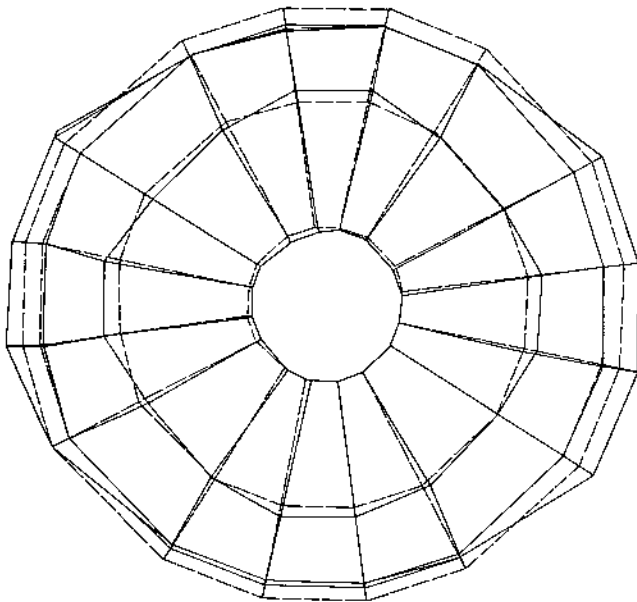


Figure 5 Displacements of operating structure, plan view

This is quite apart from any externally applied forces on the structure which we denote by F_e . (The subscripts and superscripts on F and U denote those components of F and U corresponding to the various subsets of Γ with the same superscripts).

In Ω^1 we have obtained the forces at the midpoints in the coordinate directions we can either: arrange the fluid and structure meshes so that midpoints in Ω^2 correspond to corner nodes of the mesh in Ω^2 . Or, we can modify the elements of A (the stiffness matrix for Ω^1) so that F and U may be re-interpreted as nodal forces and displacements at the corner nodes in the three global coordinate directions. This was the approach used. The force at the midpoint was split equally among the corner nodes this gives the algorithm for the modification of the A matrix.

Having performed this operation we have for the two regions:

$$\text{Region } \Omega^1 \quad F = A'U \quad (48)$$

(Fluid)

The free surface unknowns have been eliminated by the reduction process. All the unknowns are then on the interface Γ_j^1 .

For region Ω^2 (Structure):

$$\begin{bmatrix} F_e + F_I^2 \\ F_e \end{bmatrix} [K_I^2 | K^2] \begin{bmatrix} U_I^2 \\ U^2 \end{bmatrix} \quad (49)$$

where K_I^2 and K^2 are the stiffness matrices for the structure corresponding to the interface and other nodes calculated using the finite element programs. In fact the master degrees of freedom located at the crosses in Figure 1 are chosen so that the matrix equation (49) is reduced to involve only interfacial nodes. Hence we have:

$$F_e + F_I^2 = K_I^2 U_I^2 \quad (50)$$

Equations (38) and (43) may now be combined using the compatibility and equilibrium conditions (39) and (40) to give:

$$F_e = [K_I^2 + A'] U_I \quad (51)$$

This equation may be solved to give the displacements in the usual way for given forcings F_e . Because of the

compressibility of the fluid and the different phases of the forcing functions the quantities F_e , U_I and A' are all complex.

Results

The above analysis was performed for the pressure vessel, it was found that the acoustic resonance linked in with ovaling behaviour of the structure at the forcing frequency, the real parts of the displacements are shown in Figure 5. The imaginary parts were obtained, and hence the full displacement time history was available. These displacements were then applied to part of the vessel and peak stresses were obtained. A fatigue analysis was then performed on the results.

The effect of the inclusion of the fluid in the modelling reduced the displacements in the structure by 10-15%. This figure could not, however, have been deduced without performing the full analysis as the presence of the fluid has two conflicting effects. First, it introduced a damping and added mass to the surface of the shell, reducing the expected deflections. And, secondly, it enables energy to be transmitted across the shell, potentially increasing the interactions and expected deflections.

Other applications

By choosing our Green's function to represent the propagation properties of a disturbance of the fluid we can represent different kinds of fluid, the simplest being represented by the incompressible, potential Green's function.

Furthermore if we have a free surface we can choose our Green's function to satisfy the free surface boundary condition and in this way avoid the necessity of discretizing the surface. In a similar way we may easily solve external problems such as wave diffraction problems with the boundary conditions at infinity, being satisfied by our Green's function. Alternatively free surface effects may be negligible and the normal potential functions may be used.

Flow problems may also be solved by this method so long as the boundary conditions on solid surfaces are suit-

Table 1 Capabilities of the program

	With a free surface	Without a free surface
Compressible $G = \frac{e^{-ikr}(e^{i\omega t})}{-4\pi r}$ $(\nabla^2 + \kappa^2)P = 0$	1 Storage vessels	1 Acoustic noise from extended sources
	2 2-phase flow in pipes	2 Compressible flow in pipes
	3 Earthquake problems (known displacements)	3 Air duct noise propagation 4 Panel flutter
Incompressible $G = -\frac{1}{4\pi r}$ $\nabla^2 P = 0$	1 Oil storage tanks, internal and external problem	1 Simplest cases of flow
	2 Flow in pipes with free surface	2 External flow round pipes, storage tanks etc
	3 Surface waves interior problem	3 Dynamics of risers

ably modified and the right choice of physical variables made.

Table 1 illustrates some situations which are amenable to solution by this method (together with the governing equations and Green's functions associated with them).

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